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BLIND IMAGE RESTORATION BY COMBINING WAVELET TRANSFORM AND RBF NEURAL NETWORK

PING GUO* and HONGZHAI LI†

*School of Computer Science and Technology
Beijing Institute of Technology
Beijing, 100081, P. R. China

*,†Image Processing and Pattern Recognition Laboratory
Beijing Normal University, Beijing, 100875, P. R. China
*pguo@bnu.edu.cn

MICHAEL R. LYU

Department of Computer Science and Engineering
The Chinese University of Hong Kong
Shatin, Hong Kong, SAR China
lyu@cse.cuhk.edu.hk

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In this paper, we present a novel technique for restoring a blurred noisy image without any prior knowledge of the blurring function and the statistics of noise. The technique combines wavelet transform with radial basis function (RBF) neural network to restore the given image which is degraded by Gaussian blur and additive noise. In the proposed technique, the wavelet transform is adopted to decompose the degraded image into high frequency parts and low frequency part. Then the RBF neural network based technique is used to restore the underlying image from the given image. The inverse principal element method (IPEM) is applied to speed up the computation. Experimental results show that the proposed technique inherited the advantages of wavelet transform and IPEM, and the algorithm is efficient in computation and robust to the noise.

Keywords: Blind image restoration; wavelet neural network; inverse principal element method.

Mathematics Subject Classification 2000: 22E46, 53C35, 57S20

1. Introduction

Image restoration problem represents one of the primary research focuses in the field of digital image processing. Blurring is hard to avoid in any image acquisition systems, which is caused not just by only one source but by many, such as atmospheric turbulence, an out-of-focus optical system and aberrations in the imaging system, and so on. Obviously, all these blurring sources cannot be simultaneously captured exactly in a simple model. However, in general the result of these influences can be approximated by a Gaussian blur in theory,[1] so the Gaussian blur is an important study subject in image restoration for the researchers.

The existing linear image restoration algorithms assume that the Point Spread Function (PSF) is known *a priori* and attempt to reverse it in cooperation to reduce blur by utilizing available information of the PSF, original image, and noise statistics.[2] Although many studies have been done to solve this problems,[3–5] the more difficult problem faced is that the PSF is unknown in many real situations. The process of restoring an unknown image using partial or no information about the imaging system is called blind image restoration.

It is well known that the blind image restoration is a quite a challenging problem in the field of image processing, especially for those images which were degraded by Gaussian blur. The traditional method for blind image restoration is to first detect the parameters of the PSF firstly from the degraded image, and then recover the underlying image.[6,7]. However, the restoration of the Gaussian blurred image is very difficult, especially in the case where the PSF is unknown. As we know, Fourier transformation of the Gaussian function is also of Gaussian type, and it has no zero crossing point, which is the most important feature for detecting the PSF parameters, so it is hard to detect the parameters of the Gaussian function using the traditional method.

In order to solve this kind of problem, we propose the radial basis function (RBF) neural network based method, which can get better visual effects for some images,[8] but the drawback is that it time-consuming in practice. Later, the inverse principal element method (IPEM) is proposed in order to improve the computation efficiency.[9] This algorithm is relatively running faster than other neural network based algorithm but is affected by the noise.

In order to reduce the noise the wavelet transform method, that is successfully used in the fields of image compression, image segmentation image de-noising and so on, etc. are considered first.[10–13] In this paper, a new blind image restoration technique named wavelet inverse principal element method (WIPEM) is developed, which have two modules including a IPEM and a wavelet neural network. The wavelet neural network module is used to reduce noises and thus the WIPEM algorithm is robust to the noise.

2. Background

2.1. *The RBF neural network based method*

The image restoration can be considered as an "inverse problem", that is, the true images (underlying images) are degraded by various degraded systems in nature. In image restoration, we try to recover the underlying image from the observed image.

The process of digital image degradation is generally modeled as Eq. (2.1)[14]:

$$g(x,y) = H[f(x,y)] + n(x,y), \tag{2.1}$$

where, $f(x, y)$ and $g(x, y)$ are the original image and the degraded image, respectively. $n(x, y)$ is the additive noise and supposed to have zero mean. (x, y) is the coordinate of the image pixel, $0 < x,\ y < N$, here x, y, N are integers, the size of image is assumed as $N \times N$ scale. H represents the degraded system; most of time H can be regard as a linear system.

The RBF neural network has an universal approximation capability[15] and has been successfully applied to many signal and image processing problems.[16–18] A model of RBF neural network is constructed to solve the blind image restoration problem.[8] In the model of RBF neural network, the input is the coordinate and the output is the image pixel gray value. Eq. (2.1) can be rewritten as below, based on the model of the RBF neural network:

$$g(\mathbf{x}_j) = \sum_{i=1}^{N} w_i \phi_i(\|\mathbf{x}_j - \mu_i\|) + w_0, \tag{2.2}$$

where $g(\mathbf{x}_j)$, $j = 1, 2, \ldots, N$, is the network's output vector, thus the gray value of the degraded image, $\mathbf{x} = \{x_i | i = 1, 2, \ldots, N\}$, is input vector. $\mathbf{W} = [\mathbf{w_1}, \mathbf{w_2}, \ldots, w_N]^T$ is the weight between the output layer and the hidden layer, μ_i are the centers of the radial functions, $\mu_i = \mathbf{x}_j$ in this work. w_0 stands for the bias neuron and for the noise in Eq. (2.1). $\phi(x)$ stands for the basis function; here the Gaussian type function is used as basis function associated with each neuron:

$$\phi_i(\|\mathbf{x} - \mu_i\|) = \exp\left[-\frac{1}{2}(\mathbf{x} - \mu_i)^T \sum_{i}^{-1} (\mathbf{x} - \mu_i)\right], \tag{2.3}$$

where Σ_i is a covariance matrix of the Gaussian type function. Then a RBF neural network model is designed to represent the observed image. In the training process, the RBF neural network generates the smooth function that minimizes the following cost function:

$$E = \|\mathbf{\Phi W} - \mathbf{g}\|^2 + \lambda \|Pg(\mathbf{x})\|^2, \tag{2.4}$$

where vector $\mathbf{\Phi W}$ are actual network output, and P is usually a differential operator with radial symmetry, λ in this equation is a positive constant called regularization parameter, and $\mathbf{\Phi}$ now is a $N \times N$ symmetrical matrix called radial basic function matrix (RBFM)

$$\mathbf{\Phi} = \begin{pmatrix} \phi(\|x_1 - x_1\|) & \phi(\|x_2 - x_1\|) & \cdots & \phi(\|x_N - x_1\|) \\ \phi(\|x_1 - x_2\|) & \phi(\|x_2 - x_2\|) & \cdots & \phi(\|x_N - x_2\|) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(\|x_1 - x_N\|) & \phi(\|x_2 - x_N\|) & \cdots & \phi(\|x_N - x_N\|) \end{pmatrix}. \tag{2.5}$$

After establishing the relationship between RBF neural network and image restoration, the Levenberg–Marquardt (L-M) algorithm then be applied to estimate parameters and recover the underlying image at the same time.[8]

The L-M algorithm is an iterative algorithm with matrix operation and it is computationally expensive.

2.2. *The IPEM algorithm*

The IPEM simplifies the RBF-based algorithm in order to reduce the computation expensive. First of all, the signal-to-noise ratios (SNR) is supposed to be high, that is, the noise can be ignored. Then the Eq. (2.2) can be represented in matrix form as follows:

$$\mathbf{G} = \mathbf{\Phi} W. \tag{2.6}$$

Here $\mathbf{G}, \mathbf{W}$ and $\mathbf{\Phi}$ stand for the degraded image, the underlying image and the RBFM, respectively.

Assume that the radial basic function is represented in the simple form as:

$$\phi(\|\mathbf{x} - \mu\|) = \exp\left[-\frac{\|\mathbf{x} - \mu\|^2}{2\sigma^2}\right], \tag{2.7}$$

where σ is the width of the radial function, assuming it is not very large for the problem discussed in this paper. In this case, the RBFM $\mathbf{\Phi}$ is a block Toeplitz matrix, and the diagonal elements of it are

$$\phi_d = \exp\left[-\frac{d^2}{2\sigma^2}\right]. \tag{2.8}$$

In the above function, $d = \|x - \mu\| = 0, 1, 2, \ldots, N-1$, stands for Euclid distance of two image pixels.

Let some ϕ_d be zero if they are smaller than a certain small threshold value. Then only few ϕ_d here are nonzero in RBFM Φ. When there are only 4 ϕ_d nonzero, the underlying image $\mathbf{W}$ can be approximated as[9]:

$$\mathbf{W} = \frac{\mathbf{G} + (\phi_2 - \phi_3)\mathbf{F}_{1,3} * \mathbf{G} + (\phi_3 - \phi_4)\mathbf{F}_{1,5} * \mathbf{G} + \phi_4 \mathbf{F}_{1,7} * \mathbf{G}}{\phi_1 - \phi_2}. \tag{2.9}$$

Here "$*$" stands for the convolution operation, $\mathbf{F}_{1,j}, j = 3, 5, 7$, stands for the size of 3×3, 5×5, 7×7 convolution kernel matrix whose elements are all ones.

To recover the image, the IPEM does not detect the σ, but uses the iterative method based on Eq. (2.6).

$$\mathbf{G} = \mathbf{\Phi}(\sigma_1)\mathbf{\Phi}(\sigma_2)\mathbf{W} = \mathbf{\Phi}(\sigma_1 + \sigma_2)\mathbf{W}. \tag{2.10}$$

Equation (2.10) shows that the one little heavy Gaussian blur can be divided into product of two light Gaussian blur. The basic idea of IPEM is to de-blur the image with a certain Gaussian basis function, which has small blur width. Even it the function is not the "real" Gaussian basis function, the IPEM can de-blur the image a little at each iterative step. So if we de-blur the image with IPEM iteratively, the $\mathbf{W}$ will then approach the underlying image. After that point, the image would be over-recovered if we continue to deconvolute the image.

Suppose that the underlying image $\mathbf{W}$ is smooth, and the blur width is relatively small, Eq. (2.9) can be solved by iterative method. However, when the degraded image includes noise, the IPEM algorithm is not sufficient for deblurring the given image.

2.3. *The wavelet neural network*

The wavelet was reported to perform well on image de-noising.[19] In order to deal with degraded noisy image, we propose to apply the wavelet transform technique. The basic idea is to combine the wavelet and the artificial neural network (ANN) to construct the wavelet neural network (WNN).

The origin of WNN can be traced back to the work by Daugman, in which Gabor wavelets were used for image classification.[20] Wavelet networks have become popular after the work by Zhang and Benveniste.[21] They have shown that an arbitrary continuous function on a compact set can be approximated by a WNN within a finite precision. Kreinovich has proven that wavelet neural networks are asymptotically optimal approximations for functions of one variable.[22]

An interesting alternative to wavelet networks consists of using a dictionary of dyadic wavelets and to optimize only the weights. This approach is generally referred to as wave-net or wavenets. It was first proposed by Bakshi.[23]

Originally, wavenets did refer to neural networks using dyadic wavelets. In wavenets, the position and dilation are fixed and the weights are optimized by the neural network algorithm. The theory of wavenets has been generalized by Marc Thuillard to biorthogonal wavelets.[24] This extension to biorthogonal wavelets has led to the development of fuzzy wavenets.

Wavelet networks have been implemented successfully to continuous parameter WNN, frame function WNN and WNN on orthogonal basis function.[25] There are two main structures of WNN: one is incompact WNN, and the other is compact WNN.[26]

The structure of incompact WNN is shown in Fig. 1. In this WNN, the input vector will be transformed by wavelet firstly to a feature space, then processed by the ANN to get the classified or approximated result. In this work, we adopt the incompact WNN because the structure of the incompact WNN is much more simple than that of the compact WNN. Here, if the compact WNN is chosen, the computation will not be reduced. From the viewpoint of computation, the incompact WNN is better.

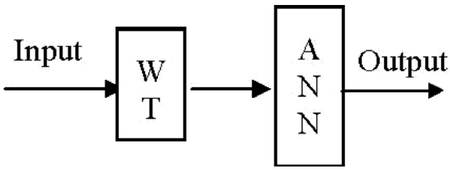


Fig. 1. The structure of incompact WNN.

3. Wavelet Inverse Principal Element Method

3.1. *The architecture of WIPEM*

The IPEM is suitable for processing a degraded image that is not noisy. However, in the real world, many images are noisy. To solve this problem, we develop the WIPEM under the framework of incompact WNN. In this method, incompact WNN is constructed to reduce the noisy image and the IPEM module is used to restore the underlying image.

In recent years there has been a fair amount of research on wavelet thresholding and threshold selection for signal de-noising.[13,28] The wavelet analysis technique provides an appropriate basis for separating noisy signal from the image signal, then under the framework of WNN the noise can be reduced well. The principle of noise reduction with wavelet is that the wavelet transform is good at energy compaction; in the signal wavelet representation, the small wavelet coefficients are more likely related to noise and large wavelet coefficients related to important signal features.[27] These small wavelet coefficients can be omitted without affecting the significant features of the image. That is so-called wavelet thresholding method, which is a traditional and effective image de-noise method.

The architecture of the WIPEM is shown in Fig. 2. In the figure, $\mathbf{G}, \mathbf{G}_g, \mathbf{G}_h$ are degraded image, the high frequency part of the image and the low frequency part of the image, respectively. $\mathbf{F}, \mathbf{F}_g, \mathbf{F}_h$ is the restored image, the high frequency part of the image and the low frequency part of the image, respectively.

3.2. *Algorithm*

Under the framework of WNN, the steps to implement the WIPEM algorithm are shown below.

Step 1: To decompose the degraded image $\mathbf{G}$ using the wavelet transforms, we can obtain

$$\mathbf{G} = \mathbf{G_{h1}} + \mathbf{G_{gH1}} + \mathbf{G_{gV1}} + \mathbf{G_{gD1}}, \tag{3.11}$$

where the decomposed sub-matrix $\mathbf{G_{h1}}$ stands for the low frequency part, the $\mathbf{G_{gH1}}$, $\mathbf{G_{gV1}}$, $\mathbf{G_{gD1}}$ (horizontal, vertical and diagonal) stands for the high frequency part of the image. The size of all decomposed matrices is the same, whose scale is $L \times L$, $L = N/2$. Then we get the high frequency and low frequency parts of the degraded image, and these images are relatively small compared with original images.

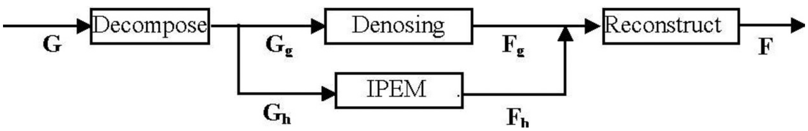


Fig. 2. The architecture of the WIPEM.

Step 2: In the high frequency parts, the noise is reduced using the wavelet thresholding method. And at the same time, the low frequency part is deblurred using the IPEM.

Generally speaking, natural noises can be regarded as random signals, so the underlying image is much smoother than the degraded noisy image. In frequency domain, the degraded image has much more high frequency information than the underlying image has. We can consider that the $\mathbf{G_g}$ contain nearly all noise and the $\mathbf{G_h}$ is nearly noise-free. Thus applying the WIPEM algorithm can de-noise and de-blur the given degraded image at the same time.

Step 3: The final restored image is reconstructed by those de-noised high frequency parts and the de-blurred low frequency part.

$$\mathbf{F} = \mathbf{F_{h1}} + \mathbf{F_{gH1}} + \mathbf{F_{gV1}} + \mathbf{F_{gD1}}, \tag{3.12}$$

where the matrix $\mathbf{F_{h1}}$ stands for the de-blurred image of low frequency part, the $\mathbf{F_{gH1}}$, $\mathbf{F_{gV1}}$, $\mathbf{F_{gD1}}$ stands for the de-noised results of high frequency part of the degraded image.

4. Experiments and Discussions

To verify the proposed WIPEM algorithm, we conduct two experiments as follows.

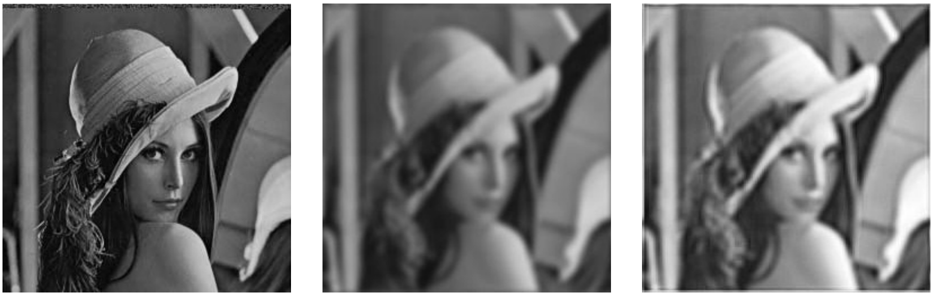
4.1. *Experiments*

In the experiments, we make the following assumptions about the image process: The image degradation is described by the linear model of Eq. (2.1). The form of PSF is assumed to be of Gaussian function type, but the σ parameter is unknown. As we know, the task of blind restoration is achieved by using partial information about the imaging process as a reference to recover the underlying image and PSF from the blurred image. Here, we only assume the parametrical PSF model; other explicit knowledge of either original image or true PSF is not required.

The first experiment is to restore the degraded image without noise. The Lenna [Fig. 3(a)] was blurred by Gaussian type blur source, as shown in Fig. 3(b). Figure 3(c) is the result of applying WIPEM algorithm, From the result, we can see that the blur is reduced, and no ringing effect in the restored image is observed; that means the algorithm has good stability.

The other experiment examines how the WIPEM algorithm deals with a blured image with additive noises. A Gaussian noise was added to the blurred image, shown as Fig. 4(a). In this case, it is very hard to get better visual effect in restoring degraded noisy image with the IPEM algorithm only. The result obtained when applying WIPEM algorithm to the blurred noisy image is shown in Fig. 4(b).

Figure 5 demonstrates the experiment with a "pepper" image. In this experiment, we also compare the effectiveness of WIPEM algorithm with deconvblind



(a) (b) (c)

Fig. 3. Example of blind image restoration by IPEM. (a) The original Lenna image, (b) the blurred image, and (c) the restored image.

(a) (b) (c)

Fig. 4. Example of blind image restoration by WIPEM algorithm. (a) The blurred Lenna image, Gaussian type noise was added, (b) the restored image with WIPEM algorithm, and (c) the restored image using deconvblind function in Matlab image processing toolbox.

algorithm in Matlab image processing toolbox. Similar to Lenna image experiment, we blur the original pepper image and add the Gaussian noise to obtain degraded image, which is shown in Fig. 5(b). Then applying deconvblind algorithm we obtain the restored result shown as Fig. 5(c). Figure 5(d) shows the result obtained by using WIPEM algorithm. It is obvious that visual effect of image in Fig. 5(d) is better than that in Fig. 5(c).

We did these experiments with Matlab software. There are some classical methods for image restoration in the Matlab image processing toolbox. Here, we use the function "deconvblind" to restore the degraded image with true PSF. The result is shown as Fig. 4(c).

Comparing Fig. 4(b) with Fig. 4(c), the experiment results shows that the WIPEM algorithm can reduce the noise and recover image well. It can be observed that there are artifacts in the restored image by the function "deconvblind", but no artifacts is observed in Fig. 4(b). This illustrates that the WIPEM algorithm is more stable than the classical method as evidenced by the wonderful quality of

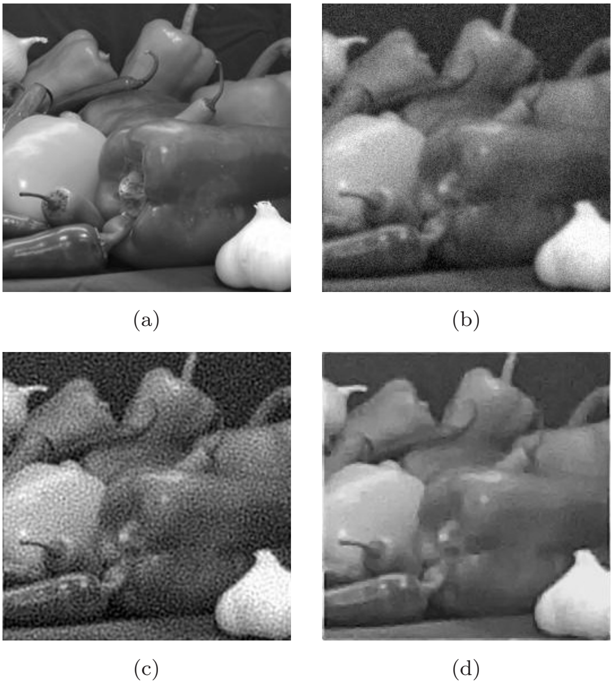


Fig. 5. Example of blind image restoration by WIPEM algorithm. (a) The original pepper image, (b) the blurred pepper image where Gaussian type noise was added, (c) the restored image using deconvblind function in matlab image processing toolbox, and (d) the restored image with WIPEM algorithm.

the wavelet multi-scale analysis capability. The experiment with "pepper" image shown in Fig. 5 also illustrates this conclusion.

We also do the experiments with some other images. For some images, the effect of the de-blurring is no obvious different compared with the "deconvblind" method. One reason is that the true PSF is used in the "deconvblind" method, which has more information and is not a blind restoration. But, in the WIPEM algorithm, we iteratively approximate the true PSF and little prior information is required. Another reason is the RBFM is simplified, some information about the RBFM is left out.

4.2. *Discussions*

There are some advantages for the proposed framework of combination of the incompact WNN and the IPEM algorithm.

The IPEM algorithm is a RBF neural network-based method; when the size of input vector is N^2, the degree of the computing complexity is the order of N^4 if we use matrix process. After we decompose the degraded image into 4 sub-matrices, the low frequency part of the image is only $1/4$ times as big as the degraded image. That is, the input vector in WIPEM is one quarter as big as original image. Experiments show that the WIPEM algorithm is more efficient than that of original IPEM in both computing time and storage space.

Secondly, the WIPEM algorithm is much more robust to noise. If only the IPEM algorithm is applied, the noise effect will be magnified when deblurring the degraded image. In the WIPEM algorithm, most of the noises are reduced in the high frequency part by the WNN module. Also, the low frequency part is much smoother than the degraded noisy image, so the IPEM module will not be infected by the noise and good results are expected. Experiment results proved our analysis.

5. Conclusions and Further Work

In this paper, under the framework of WNN, the WIPEM algorithm has been proposed as a new blind image restoration method, which combines wavelet transform and RBF neural network-based technique. Experimental results show that the WIPEM algorithm is a fast, robust algorithm compared to the previous IPEM algorithm. However, we should mention that in the noiseless environment, the WIPEM algorithm could not get better result than that of the IPEM. Our further work will try to improve the proposed algorithm in order to obtain better visual restoration effects.

There may be possibly three ways to improve the algorithm. Firstly, the IPEM algorithm itself should be improved. That is, let more ϕ_d be non-zero in Eq. (2.9), then the true PSF can be approximated by the RBFM with higher accuracy. Secondly, the compact WNN may be considered and utilize most advantages of WNN. Finally, the more suitable time to stop the iterative process of the IPEM should be found. In the IPEM, the stop time is determined by the error between $\mathbf{G}$ and $\mathbf{\Phi W}$. But in WIPEM, the resolution of the image is actually half of the previous one. The algorithm thus should be changed on a count of this difference in the future.

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