This homework is optional. If you turn it in for grading, you can use it to replace your lowest grade from the other five homeworks (provided that you followed the homework guidelines throughout the semester).

Each of the questions is worth 10 points. Please turn in solutions to *four* questions of your choice. Write your name, your student ID, and your TA's name on the solution sheet.

Please write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions and list your collaborators on your solution sheet. Copying someone else's solution will be considered plagiarism and may result in failing the whole course.

Questions

1. How many 5-card poker hands are there in which:
   (a) There are exactly two suits present in the hand?
   (b) There are no three cards of the same suit?
   (c) There are more aces than cards of any other face value?

2. How many seating arrangements are there with six boys and six girls if
   (a) The table is square with 3 seats on each side. (Two arrangements are identical if they differ by a turn of the table that keeps the chairs in place).
   (b) The table is round, the 12 seats are equally spaced, and every boy must sit across from a girl.
   (c) The table is square (with 3 seats on each side) and at least one girl must sit on each side.

3. Use the pigeonhole principle to prove the following propositions.
   (a) Thirty thousand refugees from eleven countries reached Europe last month. They are to be resettled into five nations. At least one host nation must take in at least 500 refugees from the same country of origin.
   (b) Among every 25 points inside a regular hexagon of side length 1 there is a pair within distance 1/2.
   (c) Every set S of 10 integers between 0 and 42 contains four distinct elements x, y, z, w such that 43 divides $x - y + z - w$.

4. Use the inclusion-exclusion principle to answer the following questions:
   (a) Is it possible to have 18 students and some number of clubs so that each club has at most 7 students, every two clubs have at least two students in common, but no three clubs have a student in common?
   (b) In how many ways can you partition the integers between 1 and 100 into 3 nonempty sets? (**Hint:** Calculate the size of the complement.)

5. Let X be the set of bit sequences of length $n - k$ with exactly k zeros and Y be the set of n bit sequences that have k zeros no two of which are in consecutive positions and that always end in a 1. For example, if $n = 7$ and $k = 3$ then $(0, 1, 1, 0, 1, 0, 1) \in Y$ but $(0, 1, 1, 1, 0, 0, 1), (0, 1, 0, 1, 0, 1, 0) \notin Y$.

   Let $f\colon X \to Y$ be the function that does the following operation to its input bits from left to right: (1) If the input bit is a 1, output a 1. (2) If the input bit is a 0 output the pair 0, 1. For example, $f(0, 1, 0, 0) = (0, 1, 1, 0, 1, 0, 1)$.

   (a) Show that f is injective.

   (b) Show that f is surjective.

   (c) How many bit sequences are there with z zeros, o ones in which there is no pair of consecutive zeros?

   (d) (**Extra credit**) How many circular arrangements of z zeros and o ones are there in which there is no pair of consecutive zeros?

6. In this question you will show that it is possible to have a collection of as many as 430 million people that does not contain a group of 50 friends or a group of 50 strangers.

   (a) How many possible friendship patterns are there for a collection of $N = 430$ million people?

   (b) Show that the number of friendship patterns that do not contain a group of 50 friends is at most $\binom{N}{50}2^{-1225}$ of the total number of friendship patterns from part (a). (**Hint:** Fix a set F of 50 friends and count the number of patterns in which every two people in F must be friends.)

   (c) Show that $\binom{N}{50}2^{-1224} < 1$ and explain why the above claim follows. (The bounds for $\log n!$ from Lecture 9 may come in handy. If you use a computer, explain your program.)