Each of the questions is worth 10 points. Please turn in solutions to *four* questions of your choice. Write your name, your student ID, and your TA's name on the solution sheet.

Please write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions and list your collaborators on your solution sheet. Copying someone else's solution will be considered plagiarism and may result in failing the whole course.

Questions

1. Prove the following propositions about graphs.

   (a) If the degree of every vertex is at least two, then the graph has at least one cycle.

   (b) If every connected component has a cycle, then the graph has at least as many edges as vertices.

   (c) If the graph is connected but not bipartite, then there is a walk of even length and a walk of odd length between every pair of vertices. (In a walk the vertices need not be distinct.)

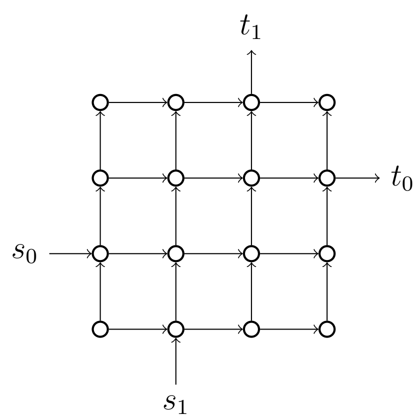
2. Consider the following stable matching scenarios with n boys and n girls.

   (a) Give preference lists for $n = 3$ in which all of these matchings are stable:
   - Every boy gets his first choice and every girl gets her last choice;
   - Every girl gets her first choice and every boy gets his last choice;
   - Everyone gets their second choice.

   (b) Show that when $n = 3$, for any preference lists there exists at least one boy-girl matching that is *not* stable.

   (c) For every n show that if all the boys' preference lists are the same and all the girls' preference lists are the same, then there is exactly one stable matching.

3. Let $n \geq 3$ be two integers. Consider the following directed graph G. The vertices of G consist of the grid points (i, j), $1 \leq i, j \leq n$ together with the four special elements s_0, t_0, s_1, t_1. The edges of G are all edges of the form $((i, j), (i + 1, j))$ and $((i, j), (i, j + 1))$ together with the four edges $(s_0, (1, 2)), (s_1, (2, 1)), ((n, n - 1), t_0), ((n - 1, n), t_1)$. Here is a diagram for $n = 4$:



(a) Show that for all $n \geq 3$, G has no cycles. (One way to show this is to give a topological sort of G with a proof that it is a topological sort.)

(b) Show that for all $n \geq 3$, G is not a switching network.

4. Consider the following directed acyclic graph G. The vertices of G are the integers from 1 to 20. The pair (u, v) is an edge of G if (and only if) u divides v and $u \neq v$.

   (a) Show that G has a path of length 4 but no path of length 5.

   (b) An *antichain* is a subset A of vertices so that there is no edge between any two vertices in A.[1] Show that G has an antichain of size 10.

   (c) Show that there exist 10 vertex-disjoint paths in G that cover all 20 vertices.

   (d) Show that G has no antichain of size 11.

   (e) **(Extra Credit)** If the vertices of G were the integers from 1 to n what would be the longest path and the largest antichain in G? Provide a proof.

5. Say vertex u is *reachable* from vertex v in at most ℓ steps if there exists a path from u to v of length at most ℓ. You are given a connected graph G with n vertices in which the degree of every vertex is at most d, where $d \geq 2$.

   (a) Let v be a vertex in G. Use induction to prove that for every $\ell \geq 0$, the number of vertices that are reachable from v in at most ℓ steps is strictly less than $d^{\ell+1}$.

   (b) Use part (a) to show that there exists two vertices u, v in G such that every path from u to v has length at least $\lfloor \log_d n \rfloor$. ($\lfloor x \rfloor$ is the largest integer not larger than x, e.g., $\lfloor 4.21 \rfloor = \lfloor 4 \rfloor = \lfloor 4.7 \rfloor = 4$.)

   (c) **(Extra credit)** Show that if every vertex in G has degree exactly d, then for every pair of vertices u, v in G, there is a path from u to v of length at most $3n/d$.

6. This question concerns spanning trees of the hypercube H_n which was defined in question 6 of Homework 3.

   (a) Give a spanning tree for H_3 that contains no path of length 6.

   (b) Show that H_n has a spanning tree that contains no path of length $2n$. (**Hint:** Use induction. You may need to prove a stronger proposition.)

   (c) Show that for all $n \geq 2$, every spanning tree of H_n contains a path of length $n + 1$.

   (d) **(Extra credit)** Does every spanning tree of H_n contain a path of length $2n - 1$?

[1] Antichains are only defined for *transitive* digraphs: if (u, v) and (v, w) are edges, so is (u, w). The digraph G is transitive.