CSCI 5160: Approximation Algorithms
The Chinese University of Hong Kong, Spring 2017

Homework 1
due Thursday March 16 in class

Collaborating on homework and consulting references is encouraged, but you must write your own solutions in your own words, and list your collaborators and your references. Copying someone else's solution will be considered plagiarism and may result in failing the whole course.

Please answer clearly and concisely. Explain your answers.

(1) Suppose you are given a sequence of nonnegative real numbers $d_{i,j}$ $(1 \leqslant i < j \leqslant n)$, and you want to know whether there are points $v_1, \ldots, v_n$ in the n-dimensional Eucldean space such that their pairwise distances are exactly $d_{i,j}$ (that is, $\|v_i - v_j\|_2 = d_{i,j}$ for all $1 \leqslant i < j \leqslant n$).

Formulate this problem as the feasibility of a semidefinite program.

(2) (a) Given real numbers $a_1, \ldots, a_n$ as parameters, consider the following linear program P (whose variables are $x_1, \ldots, x_n$ and y):

$$\begin{aligned} \min \sum_{1 \leqslant i \leqslant n} x_i & \\ a_i - y \leqslant x_i & \qquad \forall i \\ y - a_i \leqslant x_i & \qquad \forall i \\ x_i \geqslant 0 & \qquad \forall i \end{aligned}$$

In an optimal solution, what is y (in terms of $a_1, \ldots, a_n$)?

(b) Write down the dual to the above linear program (use λ_i as the dual variable for the constraint "$a_i - y \leqslant x_i$" and μ_i as the dual variable for the constraint "$y - a_i \leqslant x_i$").

(c) Let D denotes your answer to the previous part (so D is the dual to P). Suppose one modifies D' as follows to obtain a new linear program D':

i. replace the constraint "$\sum_i \mu_i = \sum_i \lambda_i$" in D with "$\sum_i \mu_i = 4 \sum_i \lambda_i$"

ii. change the objective function to "$\sum_i (4\lambda_i - \mu_i) a_i$"

What is the dual program P' of D'? In an optimal solution of P', what is y (in terms of $a_1, \ldots, a_n$)?

Hint: look up "percentile".

(3) Consider the maximum flow problem from s to t on a directed graph, where each edge has capacity one. A fraction s-t flow solution with value k is an assignment of each edge e to a fractional value $x(e)$ satisfying

$$\sum_{e \in \delta^{\text{out}}(s)} x(e) = \sum_{e \in \delta^{\text{in}}(t)} x(e) = k,$$

$$\sum_{e \in \delta^{\text{in}}(v)} x(e) = \sum_{e \in \delta^{\text{out}}(v)} x(e) \qquad \forall v \in V \setminus \{s, t\}$$

$$0 \leqslant x(e) \leqslant 1 \qquad \forall e \in E.$$

In the above, $\delta^{\text{out}}(v)$ denotes the set of out-going edges from v, and $\delta^{\text{in}}(v)$ denotes the set of incoming edges to v.

Use the multiplicative weight update algorithm to find an approximation solution to the above linear program by reducing the flow problem to the problem of finding shortest paths between s and t. Analyze the convergence rate and the total complexity of your algorithm to compute a flow of value at least $k(1-\varepsilon)$ for any $\varepsilon > 0$.

(4) Compute all the eigenvalues of the (unnormalized) adjacency matrix of the hypercube graph H_d. Also specify the multiplicities of these eigenvalues.

The hypercube H_d has 2^d vertices that are identified with binary strings of length d. Let $\{0,1\}^d$ denote the set of such strings. Two different vertices $x, y \in \{0,1\}^d$ are adjacent if they agree at $d-1$ positions (and differ at the remaining position).

Hint: First guess a nice eigenbasis for the adjacency matrix.

(5) Prove the following strengthening of Alon–Milman inequality by modifying the proof given in class: For any d-regular graph G,

$$\phi(G) \leqslant \sqrt{(2-\lambda_2)\lambda_2},$$

where λ_2 is the second smallest eigenvalue of the normalized Laplacian $\mathcal{L}$ of G.

Hint: $\sum_{(i,j)\in E}(y_i+y_j)^2 + \sum_{(i,j)\in E}(y_i-y_j)^2 = 2d\sum_i y_i^2$.