ENGG2440A variant

1. Show that if x is irrational and y is any real number then at least one of $x + y$ and $x - y$ must be irrational.

   Solution: We prove the contrapositive. Assume $x + y$ and $x - y$ are both rational. Then so is $\frac{1}{2}(x + y) - \frac{1}{2}(x - y) = x$.

2. A box contains 100 black balls and 99 white balls. In each step Alice takes out two balls of the same colour and puts in one ball of the opposite colour. Can Alice ~~empty the box~~ be left with exactly one ball of each colour in the box?

   Solution: No. We show that the predicate "3 divides $b - w - 1$" is an invariant of the underlying state machine, where b and w is the number of black and white balls respectively. The invariant holds initially. We now argue that it is preserved by the transitions, so assume 3 divides $b - w + 1$ before a given transition. There are two possibilities after the transition: Either the box contains $b-2$ black and $w+1$ white balls, in which case $(b-2)-(w+1)-1 = (b-w-1)-3$ is a multiple of 3, or the box contains $b+1$ black and $w-2$ white balls, in which case $(b+1)-(w-2)-1 = (b-w-1)+3$ is also a multiple of 3. Since 3 does not divide $1 - 1 - 1 = -1$, the state in which there is exactly one ball of each colour cannot be reached.

3. Let a and b be integers. Show that if 3 is an integer combination of $2a$ and b and 5 is an integer combination of a and $2b$ then $\gcd(a, b) = 1$.

   Solution: If 3 is an integer combination of $2a$ and b, then 3 is also an integer combination of a and b. Similarly, if 5 is an integer combination of a and $2b$, then 5 is also an integer combination of a and b. Integer combinations of integer combinations are also integer combinations, so $1 = 2\cdot 3 - 5$ is also an integer combination of a and b. Since $\gcd(a, b)$ must divide all their integer combinations, $\gcd(a, b)$ divides 1, so it must be equal to 1.

4. In a group of 15 people, is it possible for each person to have exactly 3 friends? (If Alice is a friend of Bob we assume Bob is also a friend of Alice.)

   Solution: No. Suppose for contradiction this was possible. Then the sum of degrees in the friendship graph would have been $15 \cdot 3 = 45$. But the sum of the degrees equals twice the number of edges, which is an even number, contradicting the fact that 45 is odd.

5. Sort these three functions in increasing order of growth: $\sqrt{n} \cdot \log n$, $n/\sqrt{\log n}$, $\sqrt{n \cdot \log n}$. For your sorted list f, g, h show that f is $o(g)$ and g is $o(h)$.

   Solution: $\sqrt{n \log n}$ is $o(\sqrt{n} \log n)$ because the ratio $\sqrt{n \log n}/\sqrt{n} \log n$ equals $1/\sqrt{\log n}$, which eventually becomes and stays smaller than any given constant. $\sqrt{n} \log n$ is $o(n/\sqrt{\log n})$ because the ratio $\sqrt{n} \log n/(n/\sqrt{\log n})$ equals $(\log n)^{3/2}/n^{1/2}$. In Lecture 7 we showed that $(\log n)^a$ is $o(n^b)$

for any constants $a, b > 0$, so this ratio becomes and stays smaller than any constant when n is sufficiently large.

6. The vertices of graph H are the 20 integers from -10 to 10 except 0. The edges of H are the pairs $\{x, y\}$ such that $x = -y$ or $|y - x| = 1$. How many perfect matchings does H have?

   Solution: Let $f(n)$ denote the number of matching of the analogous graph H_n with $2n$ vertices in which the integers -10 and 10 are replaced by $-n$ and n. There are two possible ways in which vertex n can be matched: Either it is matched to $-n$, in which case the remaining vertices to be matched induce the graph H_{n-1}, or it is matched to $n-1$, in which case $-n$ must also be matched to $-(n+1)$ and the remaining vertices to be matched induce the graph H_{n-2}. Therefore the number of matchings $f(n)$ satisfies the recurrence $f(n) = f(n-1) + f(n-2)$ for all $n \geq 2$. By inspection we have that $f(0) = 1$ and $f(1) = 1$. This is exactly the same recurrence we had in Lecture 7 and we can calculate the following values for $f(n)$ when $n \leq 10$:

n	0	1	2	3	4	5	6	7	8	9	10
$f(n)$	1	1	2	3	5	8	13	21	34	55	89

   so $f(10) = 89$.

7. How many length 5 passwords are there that contain at least one digit ($\texttt{0}, \texttt{1}, \ldots, \texttt{9}$), at least one `*`, and at least one `#`? No other symbols are allowed.

   Solution: Let A, B, and C denote the sets of length 5 strings (with the given symbols) that contain no digits, no `*`, and no `#`, respectively. The set of passwords is the complement of the set $A \cup B \cup C$, so there is a total of $12^5 - |A \cup B \cup C|$ passwords. By inclusion-exclusion,

   $$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 2^5 + 11^5 + 11^5 - 1^5 - 1^5 - 10^5 + 0 \end{aligned}$$

   because the set A is the product set $\{*, \#\}^5$, the set $B \cap C$ is the product set $\{\texttt{0}, \texttt{1}, \ldots, \texttt{9}\}^5$, and so on. So the number of passwords is

   $$12^5 - 2^5 - 2 \cdot 11^5 + 2 + 10^5 = 26,700.$$

8. Prove that every tree can have at most one perfect matching.

   Solution: The proof is by strong induction on the number of vertices. If a tree has one vertex then it has no perfect matching so the proposition holds. Now assume it is true for all trees with fewer than n vertices and consider any tree T with n vertices. T must have a vertex v of degree one. This vertex v can be matched in at most one way to its unique neighbor w. We now argue that there exists at most one matching that covers all remaining vertices. The graph G obtained by removing v and w from T with all their incident edges is a forest. By the inductive assumption, each connected component of G can have at most one perfect matching, so G itself, and therefore T also, can have at most one perfect matching.

   An alternative proof is to argue the contrapositive: A union of any two distinct perfect matchings Ξ_0 and Ξ_1 on the same set of vertices must contain a cycle, so Ξ_0 and Ξ_1 cannot both be perfect matchings of a tree. (Distinct does not mean *disjoint*: Ξ_1 and Ξ_2 may share some edges.) To prove this, let v_1 be any vertex that is matched differently in Ξ_0 and Ξ_1 and v_0 be its match in Ξ_0. Consider the sequence of vertices $v_0, v_1, v_2, v_3, \cdots$ where v_2 is v_1's match in Ξ_0, v_3 is v_2's match in Ξ_1, v_4 is v_3's match in Ξ_0, and so on; the matchings alternate as vertices are added. At some point a repeated vertex $v_j = v_i$ with $j > i$ must appear in the sequence. We now argue that $j \neq i + 2$, so $v_i, v_{i+1}, \ldots, v_{j-1}$ is the desired cycle. In fact, for every $i \geq 0$, v_i and v_{i+2} must be distinct. We can prove this by induction on i: this is true when $i = 0$ by the assumption on v_1, and given that v_i and v_{i+2} are distinct, their matches v_{i+1} and v_{i+3} must also be distinct.

ESTR 2004 variant

4. What is $1^4 + 2^4 + \cdots + 99^4 \mod 10$?

   Solution: We group the terms as follows. All congruences are modulo 10.

   $$1^4 + 2^4 + \cdots + 99^4 \equiv (0^4 + 10^4 + \cdots + 90^4) + (1^4 + 11^4 + \cdots + 91^4) + \cdots + (9^4 + 19^4 + \cdots + 99^4)$$

   All terms in the k-th bracket (starting our count at zero) are congruent to k^4. As there are ten such terms in each bracket, each bracket is congruent to zero, so the whole expression is zero.

8. Show that if p and q are polynomials of degree exactly 2 over $\mathbb{F}_{17}$ then there exist integers x and y such that $p(x) \equiv q(y) \pmod{17}$.

   Solution: A polynomial of degree exactly 2 cannot take any value more than twice: Assume for contradiction that p is a degree polynomial and $p(x)$ takes the value c at three different inputs x. Then $p(x) - c$ is a degree two polynomial with three zeros, a contradiction.

   By the pigeonhole principle, it follows that every polynomial of degree exactly two must take at least nine distinct values in $\mathbf{F}_{17}$. Now assume for contradiction that all values that p and q take are distinct. Then p and q take at least 18 distinct values altogether, which is impossible.