Decidability

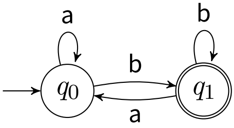
CSCI 3130 Formal Languages and Automata Theory

Siu On CHAN

Chinese University of Hong Kong

Fall 2016

Problems about automata

Does  accept input abb?

We can formulate this question as a language

$$A_{\mathsf{DFA}} = \{\langle D, w\rangle \mid D \text{ is a DFA that accepts input } w\}$$

Is A_{DFA} decidable?

One possible way to encode a DFA $D = (Q, \Sigma, \delta, q_0, F)$ and input w

```
((q0,q1)(a,b)((q0,a,q0)(q0,b,q1)(q1,a,q0)(q1,b,q1))(q0)(q1))(abb)
```

$\underbrace{\texttt{(q0,q1)}}_{Q}\underbrace{\texttt{(a,b)}}_{\Sigma}\underbrace{\texttt{((q0,a,q0)(q0,b,q1)(q1,a,q0)(q1,b,q1))}}_{\delta}\underbrace{\texttt{(q0)}}_{q_0}\underbrace{\texttt{(q1)}}_{F}\underbrace{\texttt{abb}}_{w}$

Problems about automata

$$A_{\text{DFA}} = \{\langle D, w\rangle \mid D \text{ is a DFA that accepts input } w\}$$

Pseudocode:

On input $\langle D, w\rangle$, where $D = (Q, \Sigma, \delta, q_0, F)$

Set $q \leftarrow q_0$
For $i \leftarrow 1$ to length(w)
  $q \leftarrow \delta(q, w_i)$
If $q \in F$ accept, else reject

TM description:

On input $\langle D, w\rangle$, where D is a DFA, w is a string

Simulate D on input w
If simulation ends in an accept state, accept; else reject

Problems about automata

$$A_{\mathsf{DFA}} = \{\langle D, w\rangle \mid D \text{ is a DFA that accepts input } w\}$$

Turing machine details:

Check input is in correct format
(Transition function is complete, no duplicate transitions)

Perform simulation:

```
((q̇0,q1)(a,b)((q0,a,q0)(q0,b,q1)(q1,a,q0)(q1,b,q1))(q0)(q1))(ȧbb)
((q̇0,q1)(a,b)((q0,a,q0)(q0,b,q1)(q1,a,q0)(q1,b,q1))(q0)(q1))(aḃb)
((q0,q̇1)(a,b)((q0,a,q0)(q0,b,q1)(q1,a,q0)(q1,b,q1))(q0)(q1))(abḃ)
((q0,q̇1)(a,b)((q0,a,q0)(q0,b,q1)(q1,a,q0)(q1,b,q1))(q0)(q1))(abb)̇
```

Problems about automata

$A_{\mathsf{DFA}} = \{\langle D, w\rangle \mid D \text{ is a DFA that accepts input } w\}$

Turing machine details:

Check input is in correct format
(Transition function is complete, no duplicate transitions)

Perform simulation: (very high-level)

Put markers on start state of D and first symbol of w
Until marker for w reaches last symbol:
    Update both markers
If state marker is on accepting state, accept; else reject

Conclusion: A_{DFA} is decidable

Acceptance problems about automata

$$A_{\mathsf{DFA}} = \{\langle D, w\rangle \mid D \text{ is a DFA that accepts input } w\} \quad ✓$$

$$A_{\mathsf{NFA}} = \{\langle N, w\rangle \mid N \text{ is an NFA that accepts input } w\}$$

$$A_{\mathsf{REX}} = \{\langle R, w\rangle \mid R \text{ is a regular expression that generates } w\}$$

Which of these is decidable?

Acceptance problems about automata

$$A_{\text{NFA}} = \{\langle N, w\rangle \mid N \text{ is an NFA that accepts input } w\}$$

The following TM decides A_{NFA}:
On input $\langle N, w\rangle$ where N is an NFA and w is a string

Convert N to a DFA D using the conversion procedure from Lecture 3
Run TM M for A_{DFA} on input $\langle D, w\rangle$
If M accepts, accept; else reject

Conclusion: A_{NFA} is decidable ✓

Acceptance problems about automata

$$A_{\mathsf{REX}} = \{\langle R, w\rangle \mid R \text{ is a regular expression that generates } w\}$$

The following TM decides A_{REX}

On input $\langle R, w\rangle$, where R is a regular expression and w is a string

Convert R to an NFA N using the conversion procedure from Lecture 4

Run the TM for A_{NFA} on input $\langle N, w\rangle$

If N accepts, accept; else reject

Conclusion: A_{REX} is decidable ✓

Other problems about automata

$$\mathsf{MIN}_{\mathsf{DFA}} = \{\langle D \rangle \mid D \text{ is a minimal DFA}\}$$

$$\mathsf{EQ}_{\mathsf{DFA}} = \{\langle D_1, D_2 \rangle \mid D_1 \text{ and } D_2 \text{ are DFAs and } L(D_1) = L(D_2)\}$$

$$E_{\mathsf{DFA}} = \{\langle D \rangle \mid D \text{ is a DFA and } L(D) \text{ is empty}\}$$

Which of the above is decidable?

Other problems about automata

$$\text{MIN}_{\text{DFA}} = \{\langle D \rangle \mid D \text{ is a minimal DFA}\}$$

The following TM decides MIN_{DFA}
On input $\langle D \rangle$, where D is a DFA

Run the DFA minimization algorithm from Lecture 7
If every pair of states is distinguishable, accept; else reject

Conclusion: MIN_{DFA} is decidable ✓

Other problems about automata

$$\mathsf{EQ}_{\mathsf{DFA}} = \{\langle D_1, D_2\rangle \mid D_1 \text{ and } D_2 \text{ are DFAs and } L(D_1) = L(D_2)\}$$

The following TM decides $\mathsf{EQ}_{\mathsf{DFA}}$
On input $\langle D_1, D_2\rangle$, where D_1 and D_2 are DFAs

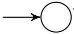
Run the DFA minimization algorithm from Lecture 7 on D_1 to obtain a minimal DFA D_1'
Run the DFA minimization algorithm from Lecture 7 on D_2 to obtain a minimal DFA D_2'
If $D_1' = D_2'$, accept; else reject

Conclusion: $\mathsf{EQ}_{\mathsf{DFA}}$ is decidable ✓

Other problems about automata

$$E_{\text{DFA}} = \{\langle D \rangle \mid D \text{ is a DFA and } L(D) \text{ is empty}\}$$

The following TM T decides E_{DFA}
On input $\langle D \rangle$, where D is a DFA

Run the TM S for EQ_{DFA} on input $\langle D,$  $\rangle$
If S accepts, T accepts; else T rejects

Conclusion: E_{DFA} is decidable ✓

Problems about context-free grammars

$A_{\mathsf{CFG}} = \{\langle G, w\rangle \mid G$ is a CFG that generates $w\}$

L where L is a context-free language

$\mathsf{EQ}_{\mathsf{CFG}} = \{\langle G_1, G_2\rangle \mid G_1, G_2$ are CFGs and $L(G_1) = L(G_2)\}$

Which of the above is decidable?

Problems about context-free grammars

$$A_{\mathsf{CFG}} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates } w\}$$

The following TM V decides A_{CFG}

On input $\langle G, w\rangle$, where G is a CFG and w is a string

Eliminate the ε- and unit productions from G

Convert G into Chomsky Normal Form G'

Run Cocke–Younger–Kasami algorithm on $\langle G', w\rangle$

If the CYK algorithm finds a parse tree, V accepts; else V rejects

Conclusion: A_{CFG} is decidable ✓

Problems about context-free grammars

L where L is a context-free language

Let L be a context-free language
There is a CFG G for L

The following TM decides L
On input w

Run TM V from the previous slide on input $\langle G, w \rangle$
If V accepts, accept; else reject

Conclusion: every context-free language L is decidable ✓

Problems about context-free grammars

$\mathsf{EQ}_{\mathsf{CFG}} = \{\langle G_1, G_2\rangle \mid G_1, G_2 \text{ are CFGs and } L(G_1) = L(G_2)\}$
is not decidable ✗

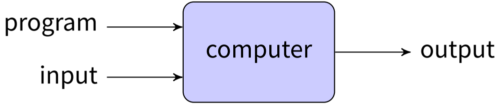
What's the difference between $\mathsf{EQ}_{\mathsf{DFA}}$ and $\mathsf{EQ}_{\mathsf{CFG}}$?

To decide $\mathsf{EQ}_{\mathsf{DFA}}$ we minimize both DFAs

But there is no method that, given a CFG or PDA, produces a unique equivalent minimal CFG or PDA

Universal Turing Machine and Undecidability

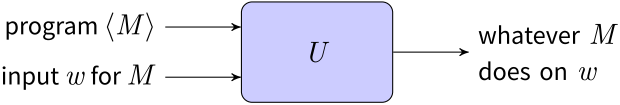
Turing Machines versus computers



A computer is a machine that manipulates data according to a list of instructions

How does a Turing machine take a program as part of its input?

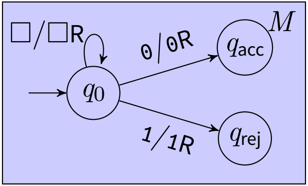
The universal Turing machine



The universal TM U takes as inputs a program M and a string x, and simulates M on w

The program M itself is specified as a TM

Turing machines as strings



A Turing machine is
$(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})$

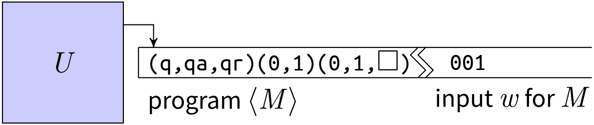
This Turing machine can be described by the string

$\langle M \rangle =$ `(q,qa,qr)(0,1)(0,1,□)`

`((q,q,□/□R)(q,qa,0/0R)(q,qr,1/1R))`

`(q)(qa)(qr)`

The universal Turing machine



U on input $\langle M, w\rangle$:

Simulate M on input w
If M enters accept state, U accepts
If M enters reject state, U rejects

Acceptance of Turing machines

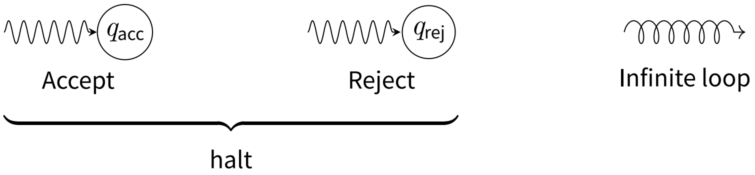
$$A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts } w\}$$

U on input $\langle M, w\rangle$ simulates M on input w

M accepts w	M rejects w	M loops on w
$\Downarrow$	$\Downarrow$	$\Downarrow$
U accepts $\langle M, w\rangle$	U rejects $\langle M, w\rangle$	U loops on $\langle M, w\rangle$

TM U recognizes A_{TM} but does not decide A_{TM}

Recognizing versus deciding



The language recognized by a TM M is the set of all inputs that M accepts

A TM decides language L if it recognizes L and halts on every input

A language L is decidable if some TM decides L