# Pushdown automata

## CSCI 3130 Formal Languages and Automata Theory

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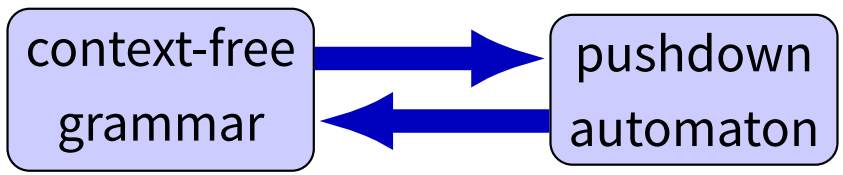
Fall 2016

# CFGs and PDAs

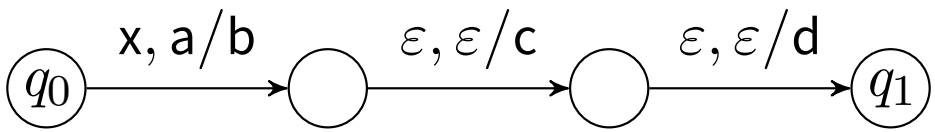
$L$ has a context-free grammar if and only if it is accepted by some pushdown automaton.



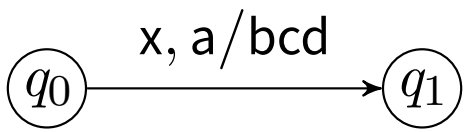

Will first convert CFG to PDA

# Convention

A sequence of transitions like




will be abbreviated as




replace a by bcd on stack

# Converting a CFG to a PDA

Idea: Use PDA to simulate derivations

$$A \to 0A1$$
$$A \to B$$
$$B \to \#$$

Example:

$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 00B11 \Rightarrow 00\#11$

Rules:

1. Write the start symbol $A$ onto the stack
2. Rewrite variable on top of stack (in reverse) according to production

| PDA control | | stack | input |
|---|---|---|---|
| write start variable | $\varepsilon, \varepsilon / A$ | $\$A$ | 00#11 |
| replace by production in reverse | $\varepsilon, A / 1A0$ | $\$1A0$ | 00#11 |

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$$A \to B$$
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Example:

$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 00B11 \Rightarrow 00\#11$

Rules:

1. Write the start symbol $A$ onto the stack
2. Rewrite variable on top of stack (in reverse) according to production
3. Pop top terminal if it matches input

| PDA control | | stack | input |
|---|---|---|---|
| write start variable | $\varepsilon, \varepsilon/A$ | $\$A$ | 00#11 |
| replace by production in reverse | $\varepsilon, A/1A0$ | $\$1A0$ | 00#11 |
| pop terminal and match | $0, 0/\varepsilon$ | $\$1A$ | 0#11 |
| replace by production in reverse | $\varepsilon, A/1A0$ | $\$11A0$ | 0#11 |
| | $\vdots$ | | |

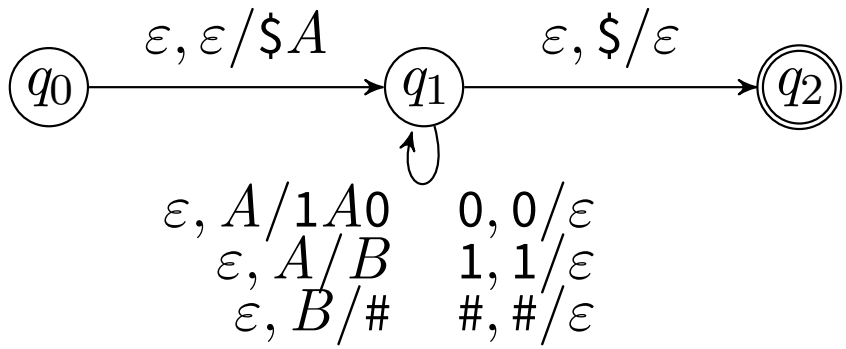
# Converting a CFG to a PDA

CFG

$A \to 0A1$

$A \to B$

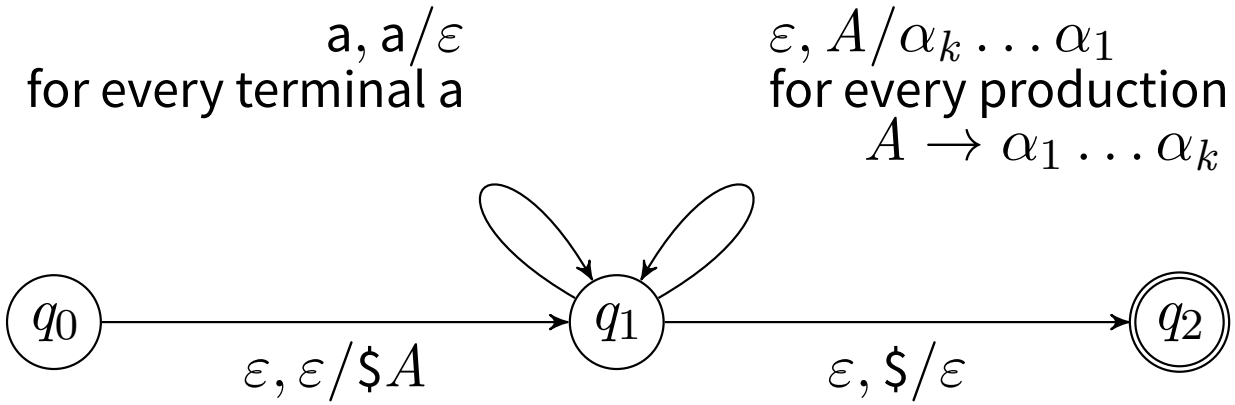
$B \to \#$



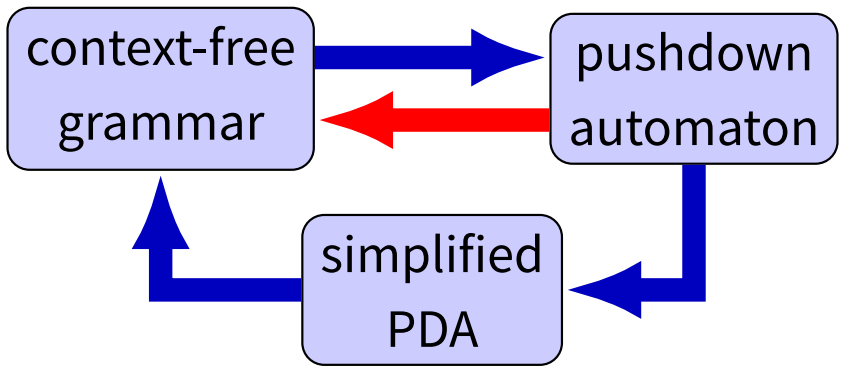

| | | | | | |
|---|---|---|---|---|---|
| input | 00#11 | 00#11 | 00#11 | 00#11 | 00#11 |
| stack | $\$A$ | $\$1A0$ | $\$1A$ | $\$11A0$ | $\$11A$ |
| | 00#11 | 00#11 | 00#11 | 00#11 | 00#11 |
| | $\$11B$ | $\$11\#$ | $\$11$ | $\$1$ | $\$$ |

$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 00B11 \Rightarrow 00\#11$$

# General CFG to PDA conversion



# From PDAs to CFGs

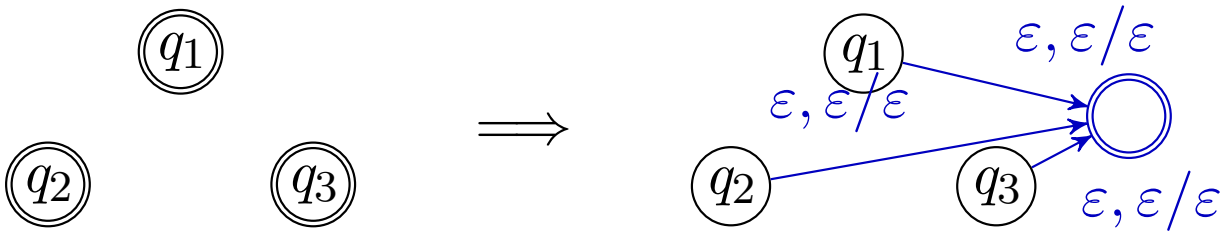



Simplified pushdown automaton:

- Has a single accepting state
- Empties its stack before accepting
- Each transition is either a push, or a pop, but not both

# Simplifying the PDA

## Single accepting state

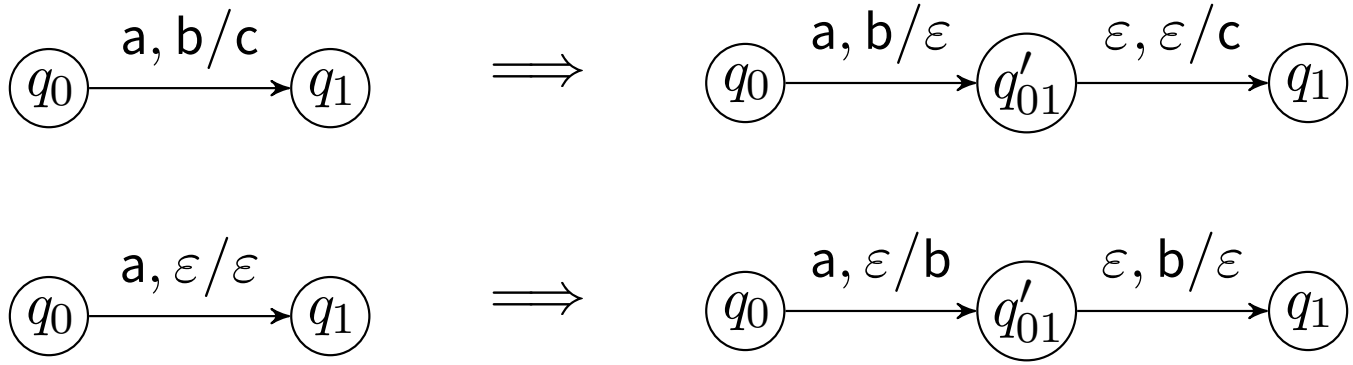



## Empties its stack before accepting

$\varepsilon$, a$/\varepsilon$ for every stack symbol a

# Simplifying the PDA

Each transition either pushes or pops, but not both

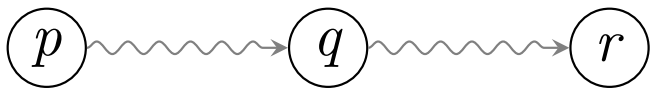
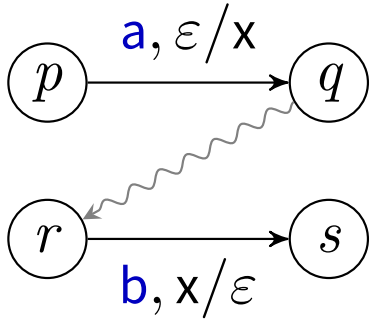


# Simplified PDA to CFG

For every pair $(q, r)$ of states in PDA, introduce variable $A_{qr}$ in CFG

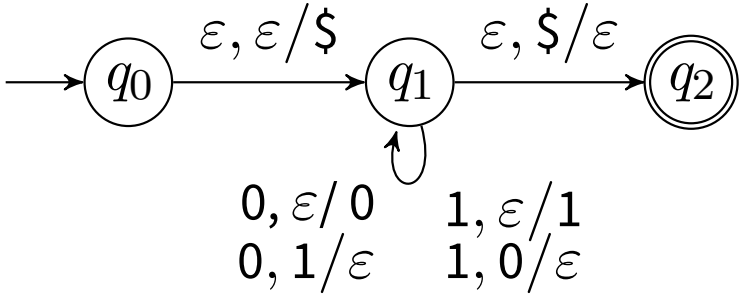
Intention: $A_{qr}$ generates all strings that allow the PDA to go from $q$ to $r$
(with empty stack both at $q$ and at $r$)

# Simplified PDA to CFG

| PDA | CFG |
|---|---|
|   | $A_{qq} \to \varepsilon$ |
|   | $A_{pr} \to A_{pq}A_{qr}$ |
|   | $A_{ps} \to \mathsf{a}A_{qr}\mathsf{b}$ <br> a $= \varepsilon$ or b $= \varepsilon$ allowed |

Start variable: $A_{pq}$ (initial state $p$, accepting state $q$)

# Example: Simplified PDA to CFG

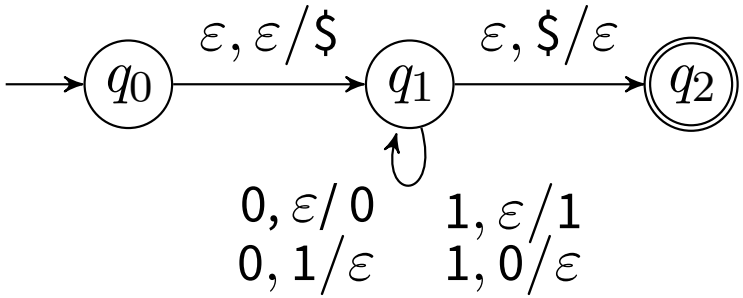



productions:

variables:

start variable:

# Example: Simplified PDA to CFG




variables: $A_{00}, A_{11}, A_{22}, A_{01}, A_{02}, A_{12}$

start variable: $A_{02}$

productions:

$$A_{02} \to A_{01}A_{12}$$

$$A_{01} \to A_{01}A_{11}$$

$$A_{12} \to A_{11}A_{12}$$

$$A_{11} \to A_{11}A_{11}$$

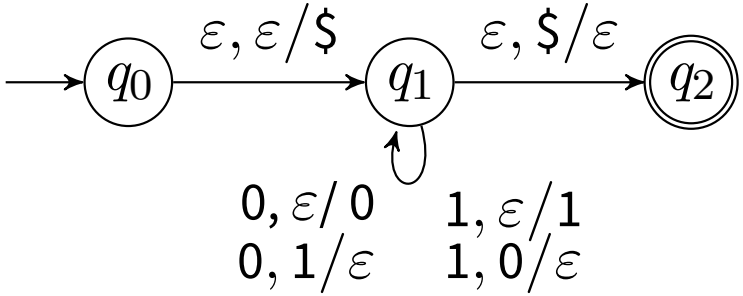
$$A_{11} \to 0A_{11}1$$

$$A_{11} \to 1A_{11}0$$

$$A_{02} \to A_{11}$$

$A_{00} \to \varepsilon,\ A_{11} \to \varepsilon,$
$A_{22} \to \varepsilon$

# Example: Simplified PDA to CFG




variables: $A_{00}, A_{11}, A_{22}, A_{01}, A_{02}, A_{12}$

start variable: $A_{02}$

productions:

$A_{02} \to A_{01}A_{12}$

$A_{01} \to A_{01}A_{11}$

$A_{12} \to A_{11}A_{12}$

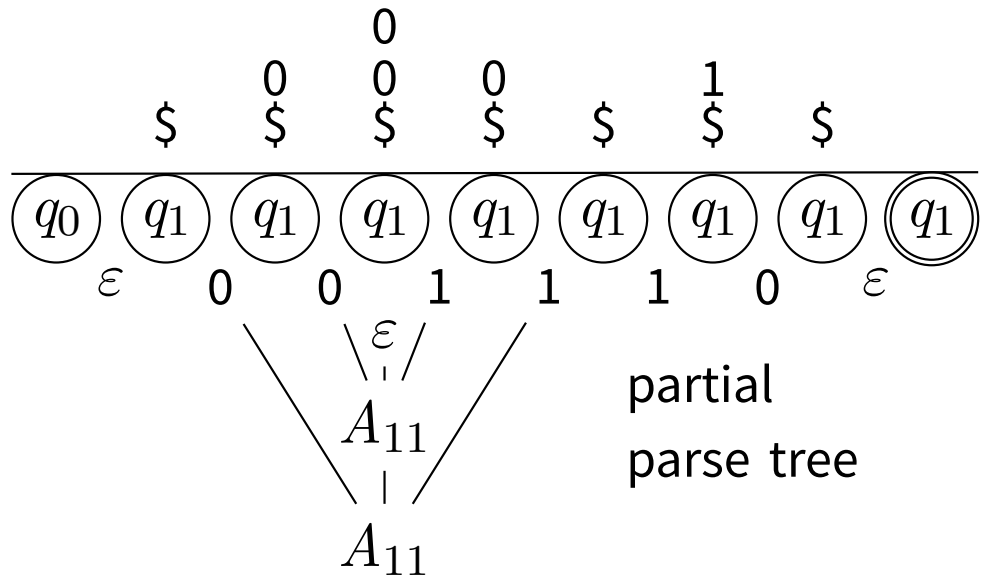
$A_{11} \to A_{11}A_{11}$

$A_{11} \to 0A_{11}1$

$A_{11} \to 1A_{11}0$

$A_{02} \to A_{11}$

$A_{00} \to \varepsilon,\ A_{11} \to \varepsilon,$
$A_{22} \to \varepsilon$




partial parse tree