Each question is worth 10 points. Explain your answers clearly.

1. You are given one sample that is either Uniform$(-1, 1)$ if $\Theta = 0$ or Uniform$(0, 4)$ if $\Theta = 1$. Your prior on Θ is equally likely $(P(\Theta = 0) = P(\Theta = 1) = 1/2)$.

   (a) What is the MAP estimator for Θ from this sample?

   (b) What is the error probability of the MAP estimator in part (a)?

2. You observe the following samples of a normal random variable with unknown mean and variance: $-1.5, -0.8, 1.9$.

   (a) What is the adjusted sample standard deviation S^2?

   (b) Give a 95% confidence interval for the actual standard deviation. Justify the use of your formula for confidence intervals.

3. A fair n-sided die with equally likely face values $1, 2, \ldots, n$ is tossed five times.

   (a) You observe the outcomes $1, 2, 1, 1, 5$. What is the maximum likelihood estimate for n?

   (b) Let MAX be the largest of the five outcomes. Is $[MAX, 2MAX]$ a 95% confidence interval for n? Justify your answer.

4. A random variable has PMF $f(-1) = f(1) = \theta$, $f(0) = 1 - 2\theta$, where θ is unknown $(0 \leq \theta \leq \frac{1}{2})$.

   (a) What is the actual standard deviation σ of the random variable?

   (b) What is the PMF of the adjusted sample standard deviation S^2 for two samples?

Each question is worth 10 points. Explain your answers clearly.

1. X is a Normal$(0, \Theta)$ random variable, where the prior PMF of the parameter Θ is $P(\Theta = 1/2) = 1/2$, $P(\Theta = 1) = 1/2$. You observe the following three independent samples of X: $1.0, 1.0, -1.0$.

   (a) What is the posterior PMF of Θ?

   (b) What is the MAP estimate of Θ?

2. The true fraction of employees in some company that support longer lunch breaks is 80%. Ten employees are polled about their support for longer lunch breaks (randomly with repetition). What is the probability that at least 70% of the polled employees support longer lunch breaks?

3. In a random 50 participant survey about favorite colors, 20 choose "blue", 15 choose "red", 10 choose "green", and 5 choose "yellow".

   (a) Give 95% confidence intervals for the popularity of blue and green among the general population using the "simplified" formula for confidence intervals.

   (b) Based on your calculation in part (a), what is your confidence level for the claim "blue is more popular than green"? Justify your answer.

4. A random variable X is Normal$(1, 1)$ with probability p and Normal$(-1, 1)$ with probability $1 - p$, where the parameter p is unknown.

   (a) What is the maximum likelihood estimator of p from a single sample X?

   (b) Is the estimator in part (a) unbiased? Justify your answer.