You are encouraged to work on this homework together as long as you write up your own solutions. If you do so, write the names of your collaborators. Please refrain from looking up solutions to the homework on the internet or in other sources. If you must, state the source.

Problem 1

Consider the following dictatorship test:

> Given a function $f\colon \{0,1\}^n \to \{0,1\}$:
>
> Apply the linearity test to f:
>
> Choose random $a, b \sim \{0,1\}^n$ and reject if $f(a) + f(b) \neq f(a+b)$.
>
> Choose random $x, y \sim \{0,1\}^n$ and a random partition (I, J) of $\{1, \dots, n\}$.
>
> If $f(x_I x_J) \neq f(x_I y_J)$ and $f(x_I x_J) \neq f(y_I x_J)$, reject.
>
> Otherwise, accept.

A random partition (I, J) of $\{1, \dots, n\}$ is chosen by including each element in I independently and uniformly at random and setting J to be the complement of I. The notation $z_I w_J$ is used for a string in $\{0,1\}^n$ whose ith coordinate is z_i if $i \in I$ and w_i if $i \in J$.

(a) Show that if f is a dictator, i.e. $f(x) = x_i$ for some $i \in \{1, \dots, n\}$, then the test accepts f.

(b) Show that if f is balanced (i.e. $\mathrm{E}_{x\sim\{0,1\}^n}[f(x)] = 1/2$) and the test accepts f with probability $1 - \delta$, then there exists a dictator x_i such that $\Pr_{x\sim\{0,1\}^n}[f(x) = x_i] = 1 - O(\delta)$. (**Hint:** Do this part under the assumption that f is a linear function first.)

Problem 2

In this problem you will argue that the following task is NP-hard: Given a system of 4LIN equations whose right hand side is always equal to one, if a $(1-\eta)$ fraction of them are simultaneously satisfiable, find an assignment that satisfies a $(1+\varepsilon)/2$ fraction of them.

Using the notation from Lecture 9, consider the following instance of 4LIN. For a random constraint (i, j) of Ψ, independent random vectors $s, s', t \in \{0,1\}^\Sigma$ and e chosen from the η-biased distribution over $\{0,1\}^\Sigma$ we include the following 4LIN equation in Ξ:

$$X_i(s)X_i(s')Y_j(t)Y_j(s_\pi + s'_\pi + \bar{t} + e) = -1.$$

Here $\bar{t}$ denotes the vector in which every entry of t is replaced by its complement.

(a) Show that if Ψ is satisfiable, it is possible to satisfy a $1 - \eta$ fraction of the equations of Ξ.

(b) Show that if (X, Y) satisfies a $(1+\varepsilon)/2$-fraction of the equations in Ξ, then for at least $\varepsilon/2$ of the pairs (i, j),

$$\sum_a (-1+2\eta)^{|a|} \hat{X}^2_{\mathrm{odd}(a)} \hat{Y}^2_a \leq -\varepsilon/2.$$

(c) Using part (b) argue that given an assignment (X, Y) that satisfies a $(1+\varepsilon)/2$-fraction of the equations of Ψ, we can efficiently sample an assignment that satisfies $\gamma = \tilde{O}(\varepsilon^2\eta)$ fraction of the constraints of Ξ in expectation.

Problem 3

Let G be a non-bipartite regular graph and $\lambda = \max(\lambda_2(G), -\lambda_n(G))$.

(a) Let $\mathbf{v}$ be a vector that assigns a $\{-1, 1\}$ value to every vertex of G. Show that

$$\left| \mathrm{E}_{(u,w)}[\mathbf{v}(u)\mathbf{v}(w)] - \mathrm{E}_u[\mathbf{v}(u)]^2 \right| \leq \lambda$$

where (u, w) are the endpoints of a random edge in G. (**Hint:** Expand $\mathbf{v}$ in the basis of eigenvectors of the normalized adjacency matrix of G.)

(b) Assume that the vertices of G are labeled by the elements of an ε-biased set $D \subseteq \{0, 1\}^n$. Let D' be the following distribution: Uniformly choose a random edge (u, w) of G and output $u + w$. Show that D' is $(\varepsilon^2 + \lambda)$-biased. (**Hint:** Think of a character function as a $\{-1, 1\}$-valued vector over the vertices of G.)

Problem 4

In this problem you will investigate how certain operations on graphs affect their eigenvalues. Assume that G is non-bipartite and regular.

(a) The *double cover* of a graph G is the graph G_2 that has two copies v_1, v_2 of every vertex v in G and edges (v_1, w_2), (v_2, w_1) for every edge (v, w) of G. For example the double cover of the 3-cycle is the 6-cycle. Show that if the eigenvalues of (the normalized adjacency matrix of) G are $\lambda_1, \ldots, \lambda_n$, then the eigenvalues of G_2 are $\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n$.

(b) **(Optional)** Let A be the adjacency matrix of G and A' be a symmetric matrix obtained by changing signs of some of the entries in A. A *2-lift* of G is the graph G'_2 that has two copies v_1, v_2 of every vertex v in G and the following edges for every edge (v, w) of G:

- If $A'_{v,w}$ is positive, then G'_2 contains the edges (v_1, w_1) and (v_2, w_2),
- If $A'_{v,w}$ is negative, then G'_2 contains the edges (v_1, w_2) and (v_2, w_1).

Show that the eigenvalues of the normalized adjacency matrix of G'_2 consist of the eigenvalues of A and the eigenvalues of A'.