In lecture 4 we learned how to do arithmetic modulo a prime number. We can use modular arithmetic to evaluate expressions like $x^2 + 3x + 1 \mod 5$ for different values of $x$. For example, when $x = 2$

$$2^2 + 3 \cdot 2 + 1 \mod 5 = 11 \mod 5 = 1.$$

Such expressions are called *polynomials*.

# 1 Polynomials in one variable

Throughout this lecture $q$, will be a prime number and we will use the symbol $\mathbb{F}_q$ to denote the set $\{0, 1, \dots, q-1\}$ of numbers modulo $q$. For any two numbers $x$ and $y$ in $\mathbb{F}_q$ we will use $x + y$, $-x$ $x \cdot y$, and $x^{-1}$ to denote the corresponding operations modulo $q$.[1] For example, for $3, 4 \in \mathbb{F}_5$, $3 + 4 = 2$ and $3 \cdot 4 = 2$.

A *polynomial* of degree $d$ over $\mathbb{F}_q$ is a function $p$ from $\mathbb{F}_q$ to $\mathbb{F}_q$ of the form

$$p(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \dots, a_{d-1}, a_d$ are numbers in $\mathbb{F}_q$ called *coefficients*. We require $a_d \neq 0$ unless $d = 0$.

Polynomials can be added and multiplied. For example, if we work over $\mathbb{F}_7$:

$$\begin{aligned}(x^2 + 3x + 6) + (x + 5) &= x^2 + 4x + 4 \\ (x^2 + 3x + 6) \cdot (x + 5) &= (x^3 + 3x^2 + 6x) + (5x^2 + x + 2) = x^3 + x^2 + 2.\end{aligned}$$

If $p$ has degree $d$ and $p'$ has degree $d'$, then their sum $p + p'$ has degree at most the larger of $d$ and $d'$ and their product $p \cdot p'$ has degree $d + d'$.

**Zeros** A value $c$ in $\mathbb{F}_q$ such that $p(c) = 0$ is called a *zero* of the polynomial $p$. For example, 1 is a zero of the polynomial $p(x) = x^3 + x^2 + x + 2$ over $\mathbb{F}_5$ because $p(1) = 0$. We can then write

$$\begin{aligned}p(x) &= p(x) - p(1) \\ &= (x^3 + x^2 + x + 2) - (1^3 + 1^2 + 1 + 2) \\ &= (x^3 - 1^3) + (x^2 - 1^2) + (x - 1) \\ &= (x-1)(x^2 + x + 1) + (x-1)(x+1) + (x-1) \\ &= (x-1)\big((x^2 + x + 1) + (x+1) + 1\big) \\ &= (x-1)(x^2 + 2x + 3)\end{aligned}$$

[1] This structure is called a *finite field*: You can add, subtract, multiply, and divide except by zero, and the usual rules of arithmetic apply.

so we get the *factorization* of $p(x)$ as $x-1$ times a polynomial of degree one lower than $p$:

$$x^3 + x^2 + x + 2 = (x-1)(x^2 + 2x + 3).$$

We can apply a similar procedure to factor a polynomial $p$ with a zero $c$ as $p(x) = (x-c)p'(x)$, where $p'$ is a polynomial of degree one lower than $p$. We *prove* the existence of this factorization a bit differently.

**Lemma 1.** *If $p$ is a polynomial of degree $d$ over $\mathbb{F}_q$ and $c$ is a zero of $p$, then there exists a polynomial $p'$ of degree $d-1$ such that $p(x) = (x-c)p'(x)$.*

*Proof.* First we do the case $c = 0$. Assume $p(x) = a_d x^d + \cdots + a_1 x + a_0$. Then $p(0) = a_0$, so $a_0 = 0$ and we can factor $p$ as

$$p(x) = a_d x^d + \cdots + a_1 x = x \cdot p'(x) \quad \text{where } p'(x) = a_d x^{d-1} + \cdots + a_1.$$

If $c$ is nonzero, look at the polynomial $p_1(x) = p(x+c)$, which also has degree $d$. Then 0 is a zero of $p_1$ so there exists a factorization $p_1(x) = x \cdot p_1'(x)$ for some $p_1'$ of degree $d-1$. Then

$$p(x) = p_1(x-c) = (x-c) \cdot p_1'(x-c) = (x-c)p'(x)$$

where $p'(x) = p_1'(x) - c$ is a polynomial of degree $d-1$ as desired. □

The *zero polynomial* is the polynomial all of whose coefficients are zero.

**Theorem 2.** *Every nonzero polynomial over $\mathbb{F}_q$ of degree $d$ has at most $d$ zeros.*

*Proof.* The proof is by induction of $d$.

**Base case $d = 0$:** A nonzero polynomial $p$ of degree 0 has the form $p(x) = a_0$ for $a_0 \neq 0$, so it doesn't have any zeros.

**Inductive step:** Assume every nonzero polynomial of degree $d-1 \geq 0$ has at most $d-1$ zeros. Let $p$ be a polynomial of degree $d$. If $p$ has no zeros, the theorem is true. Otherwise, $p$ has at least one zero $c$. By Lemma 1, $p(x) = (x-c)p'(x)$ for some polynomial $p'$ of degree $d-1$. Every other zero of $p$ is also a zero of $p'$. By our inductive assumption, $p'$ has at most $d-1$ zeros, so $p$ has at most $d$ zeros. □

A nonzero polynomial may evaluate to zero everywhere: For example, $x(x-1)(x-2) = x^3 - x$ over $\mathbb{F}_3$ evaluates to zero everywhere. However, this is impossible if the degree is smaller than $q$.

**Corollary 3.** *If $d < q$, $p$ has degree $d$, and $p(c) = 0$ for all $c \in \mathbb{F}_q$, then $p$ is the zero polynomial.*

*Proof.* Under the assumptions, $p$ has $q > d$ zeros. By Theorem 2, $p$ is the zero polynomial. □

## 2 Interpolation

One nice thing about polynomials is that we can recover the polynomial from its degree and enough of its input-output pairs. Let's see a couple of examples.

**Example 1.** Say you have a degree one polynomial $p$ over $\mathbb{F}_5$ and you are told that $p(2) = 3$ and $p(3) = 0$. What is this polynomial $p$? We know $p$ has the form $p(x) = a_1 x + a_0$ for some $a_0, a_1$ in $\mathbb{F}_5$. We are told that

$$\begin{aligned} a_1 \cdot 2 + a_0 &= 3 \quad \text{and} \\ a_1 \cdot 3 + a_0 &= 0. \end{aligned}$$

To figure out $p$, we need to solve these two equations modulo 5. If we subtract the first equation from the second one we get $a_1 = -3 = 2$. Plugging this in the first equation we get $a_0 = 3 - 2 \cdot 2 = -1 = 4$. So the desired polynomial is $p(x) = 2x + 4$.

**Example 2.** How about finding a polynomial $p$ of degree 2 over $\mathbb{F}_7$ such that $p(0) = 3$, $p(1) = 0$, and $p(4) = 3$? Again, we know $p$ has the form $p(x) = a_2 x^2 + a_1 x + a_0$ for some $a_0, a_1, a_2$ in $\mathbb{F}_7$ and we know that

$$\begin{aligned} a_0 &= 3 \\ a_2 + a_1 + a_0 &= 0 \\ a_2 \cdot 4^2 + a_1 \cdot 4 + a_0 &= 3. \end{aligned}$$

I solved this system of equations on the computer to get $a_2 = 1$, $a_1 = 3$, $a_0 = 3$, so $p(x) = x^2 + 3x + 3$.

It looks like if we are given $d + 1$ values of the polynomial, we can figure out what the polynomial is. It turns out this is always possible to do. In fact, there is a formula for this polynomial called the *Lagrange interpolation formula*. For degree one, if we are given the "data" $p(x_0) = y_0$ and $p(x_1) = y_1$, this formula tells us that $p$ is the polynomial

$$p(x) = y_0 \cdot \frac{x - x_1}{x_0 - x_1} + y_1 \cdot \frac{x - x_0}{x_1 - x_0}.$$

If you plug in $x = x_0$, the second term vanishes and the first term becomes $y_0$. If you plug in $x = x_1$, the first term vanishes and the second one becomes $y_1$ as desired. Let's re-solve Example 1 using this formula: We are told that $p(2) = 3$ and $p(3) = 0$ (over $\mathbb{F}_5$), so

$$p(x) = 3 \cdot \frac{x - 3}{2 - 3} + 0 \cdot \frac{x - 2}{3 - 2} = 2(x - 3) = 2x + 4.$$

For a degree 2 polynomial such that $p(x_0) = y_0$, $p(x_1) = y_1$, and $p(x_2) = y_2$, the Lagrange interpolation formula says that $p$ equals

$$p(x) = y_0 \cdot \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + y_1 \cdot \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + y_2 \cdot \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}.$$

If we work out Example 2 using this formula, we get

$$\begin{aligned}
p(x) &= 3 \cdot \frac{(x-1)(x-4)}{(0-1)(0-4)} + 0 \cdot \frac{(x-0)(x-4)}{(1-0)(1-4)} + 3 \cdot \frac{(x-0)(x-1)}{(4-0)(4-1)} \\
&= 6(x-1)(x-4) + 2x(x-1) \\
&= (6x^2 - 30x + 24) + (2x^2 - 2x) \\
&= 8x^2 - 32x + 24 \\
&= x^2 + 3x + 3.
\end{aligned}$$

As you can see, the formula can be cumbersome to use in practice, but it will help us prove that interpolation works.

**Theorem 4.** *Assume $d < q$. For every set of $d+1$ pairs of numbers $(x_0, y_0), \ldots, (x_d, y_d)$, all in $\mathbb{F}_q$, where $x_0, x_1, \ldots, x_d$ are all distinct, there exists exactly one polynomial $p$ of degree at most $d$ such that $p(x_i) = y_i$ for every $i$ between $0$ and $d$.*

We need to prove two things: That there is at least one $p$ (existence) and there is at most one $p$ (uniqueness).

*Proof of existence.* Let $p$ be the polynomial

$$p(x) = \sum_{i=0}^{d} y_i \cdot \prod_{j=0, j\neq i}^{d} \frac{x - x_j}{x_i - x_j} \tag{1}$$

Every term in the summation is a product of $d$ degree 1 polynomials, so the polynomial has degree at most $d$. When $x = x_i$, all but the $i$-th term in the summation contain the factor $(x - x_i)$ so they vanish. Only the $i$-th term survives and it evaluates to

$$p(x_i) = y_i \cdot \prod_{\substack{j=0 \\ j\neq i}}^{d} \frac{x_i - x_j}{x_i - x_j} = y_i,$$

so $p(x_i) = y_i$ for every $i$ between 0 and $d$. □

*Proof of uniqueness.* Assume that $p$ and $p'$ are two polynomials of degree $d$ such that $p(x_i) = y_i$ for all $i$ between 0 and $d$. We will show that $p$ and $p'$ must in fact be the same polynomial. Let $r$ be the polynomial $r(x) = p(x) - p'(x)$. The polynomial $r$ has degree at most $d$ and

$$r(x_i) = p(x_i) - p'(x_i) = y_i - y_i = 0$$

for all $i$ between 0 and $d$. So $r$ is a polynomial of degree $d$ that has at least $d+1$ zeros. By Theorem 2, $r$ is the zero polynomial, so $p$ and $p'$ must be the same polynomial. □

## 3 Secret sharing

Alice, Bob and Charlie find a stash of cash in the trash. They put it in a box for keeping until they find out who it belongs to. Each of them is afraid they will be conned by the other two, who may just decide to take the money and run. They come up with a safekeeping system: Each one will put their own lock on the outside of the box and keep the key for it. For the box to open, all three will need to agree and present their keys.

Secret sharing is the digital variant of this scenario. It involves $d + 1$ parties who want to share a secret piece of information, like the password to a shared bitcoin account. Each party $i$ gets a *share* $p(i)$, which is also some piece of information. The sharing should be done in such a way that if all $d + 1$ parties reveal their shares then they can recover the secret, but even if one refuses to cooperate the others cannot obtain any meaningful information about the secret.

It takes a bit of effort to explain all this in precise mathematical language. Instead of doing that, let me tell you somewhat informally how polynomials can be used to achieve this task. (There are also other ways to do it.)

First, we choose a large enough prime $q$ and represent the secret $s$ as a number in the set $\mathbb{F}_q = \{0, 1, \ldots, q - 1\}$. For example this web page says that $q = 5915587277$ is a 10 digit prime number. This is large enough to represent all possible secret numbers up to 9 digits long. The shares will also be numbers in $\mathbb{F}_q$.

To share the secret $s$ among $d + 1$ parties, choose a sequence of $d$ numbers $(a_1, a_2, \ldots, a_d)$ in $\mathbb{F}_q$, randomly so that the sequence is equally likely among all $q^d$ possible sequences. Set $a_0 = s$ and evaluate the polynomial

$$p(x) = a_d x^d + \cdots + a_1 x + a_0$$

At $x = 1$, 2, up to $d + 1$. Party $i$ receives the share $p(i)$.

For example, if Alice, Bob, and Charlie want to share the secret password $s = 123456789$ (and work over $\mathbb{F}_{5915587277}$), I use the computer to choose the random sequence

$$a_2 = 775093894, a_1 = 3769551523$$

which gives

$$p(x) = 775093894 \cdot x^2 + 3769551523x + 123456789.$$

I give Alice, Bob, and Charlie the shares $p(1) = 4668102206$, $p(2) = 4847348134$, and $p(3) = 661194573$, respectively.

We set up things so that the secret $s$ equals the value $p(0) = a_0$. When Alice, Bob, and Charlie want to recover $s$, they use the Lagrange interpolation formula (1) at $x = 0$ with $(x_i = i, y_i = p(i))$ for $i$ from 1 to $d + 1$:

$$p(0) = \sum_{i=1}^{d+1} p(i) \cdot \prod_{\substack{j=1 \\ j \neq i}}^{d+1} \frac{-j}{i - j}.$$

For Alice, Bob, and Charlie, this formula gives

$$\begin{aligned} p(0) &= 4668102206 \cdot \frac{(-2)(-3)}{(1-2)(1-3)} + 4847348134 \cdot \frac{(-1)(-3)}{(2-1)(2-3)} + 661194573 \cdot \frac{(-1)(-2)}{(3-1)(3-2)} \\ &= 4668102206 \cdot 3 + 4847348134 \cdot (-3) + 661194573 \cdot 1 \\ &= 123456789. \end{aligned}$$

**Security** Now we want to show if any of the $d+1$ parties refuse to cooperate, the other $d$ cannot discover any information about the secret $s$. To not discover any information does not merely mean that $d$ parties cannot *recover* the secret. They must also be unable to answer questions like "Is it an even number?" or "Does its decimal representation contain a zero"?

To reason about the power of such questions it helps to carry out a mental experiment. Imagine two scenarios: In scenario 1, the secret is $s = 123456789$. In scenario 2, the secret is $s = 100000000$. If Alice and Bob can get together and answer the question "Is the secret an even number?", they must be able to *distinguish* between these two scenarios.

How can they distinguish? They have to use the information at hand, namely their shares $p(1)$ and $p(2)$. Recall that these shares were obtained by setting $a_0$ to $s$, choosing $(a_1, a_2)$ at random, and evaluating the polynomial $p(x) = a_2x^2 + a_1x + a_0$ at $x = 1$ and $x = 2$.

The sequence of values $(p(1), p(2))$ is random; its value is only determines when we fix the choice of $a_1$ and $a_2$. In Theorem 5 we will show that regardless of the value of the secret, any pair of values $(p(1), p(2))$ in $\mathbb{F}_q \times \mathbb{F}_q$ is as likely as any other. For instance, Alice and Bob are equally likely to observe the pair of shares $(3056292003, 1111111111)$ as they are to observe the pair $(987654321, 123123123)$. Therefore the pair of values $(p(1), p(2))$ does not give any distinguishing information to tell whether the secret was 123456789 or 100000000.

**Theorem 5.** *Assume the sequence $(a_1, \ldots, a_d)$ in $\mathbb{F}_q^d$ was chosen randomly so that every possible sequence of $q^n$ values is equally likely. Then for every $a_0 \in \mathbb{F}_q$ and every $d$ distinct inputs $x_1, \ldots, x_d$, the sequence of values $(p(x_1), \ldots, p(x_d))$ takes all $q^d$ values equally likely.*

It follows that for every possible secret $s = a_0$, if party $i$ refuses to cooperate, the sequence of $d$ shares $(p(1), \ldots, p(i-1), p(i+1), \ldots, p(d+1))$ is equally likely to take on any of the $q^d$ possible values. So observing the shares of any $d$ parties does not reveal any information about the secret.

*Proof of Theorem 5.* For every choice of $a_0$, let $f\colon \mathbb{F}_q^n \to \mathbb{F}_q^n$ be the function that maps $(a_1, \ldots, a_d)$ to $(p(x_1), \ldots, p(x_d))$, where $p(x) = a_dx^d + \cdots + a_1x + a_0$. We will show that $f$ is a bijective function. Since all inputs of $f$ are equally likely, its outputs must also then be equally likely.

$f$ is surjective by the existence part of Theorem 4: For any set of values $y_1, \ldots, y_d$, there exists a polynomial $p$ of degree $d$ such that $p(x_i) = y_i$ for $i = 1$ up to $d$. $f$ maps the coefficients of this polynomial to $(y_1, \ldots, y_d)$.

$f$ is injective by the uniqueness part of Theorem 4: For every sequence of values $(y_1, \ldots, y_d)$, the polynomial $p$ whose coefficients $f$ maps to this sequence is unique. $\square$

I wrote a computer program that implements this secret sharing scheme. Feel free to play with it.

## References

The secret sharing scheme described here was first described by Adi Shamir in the article *How to share a secret?* Secret sharing is usually studied in the context of cryptography, a vast subject concerned with the study of secure communication and computation in insecure environments.