Each of the questions is worth 10 points. Please turn in solutions to *four* questions of your choice. Write your name, your student ID, and your TA's name on the solution sheet.

Please write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions and list your collaborators on your solution sheet. Copying someone else's solution will be considered plagiarism and may result in failing the whole course.

Questions

1. Find exact closed form expressions for the following sums. Explain how you discovered the expression and prove that it is correct.

   (a) $1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2$.

   (b) $(2^0 + \cdots + 2^n) + (2^1 + \cdots + 2^{n+1}) + \cdots + (2^n + \cdots + 2^{2n})$.

2. Show the following inequalities.

   (a) $2\sqrt{n+1} - 2 \leq 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n+1} - 1 - \frac{1}{\sqrt{n+1}}$.

   (b) $n^n \leq 1^1 + 2^2 + \cdots + n^n \leq (1 + \frac{1}{n-1})n^n$ (for $n \geq 2$).

   (c) $|H(n) - \ln n| = O(1)$, where $H(n)$ is the nth harmonic number.
   (**Hint:** You need to show that $H(n) \geq \ln n - O(1)$ and that $H(n) \leq \ln n + O(1)$.)

3. Sort the following functions in increasing order of asymptotic growth. (For example, if you are given the functions n^2, n, and 2^n, the sorted list would be $n, n^2, 2^n$.) Show that for every pair of consecutive functions f, g in your list, f is $o(g)$.

   (a) $n^{\log n}$, 2^n, n^n, $2^{\sqrt{\log n}}$, 2^{n^2}, e^{2^n}, n, 2^{e^n}, $(\log n)^{(\log n)^{(\log n)}}$, $n^{(\log n)^{\log \log n}}$.

   (b) The *tower function* $T(n)$ is given by the recursive formula $T(1) = 2$ and $T(n+1) = 2^{T(n)}$ for $n \geq 1$. Where do the functions $T(n)$ and $T(\log n)$ (assuming n is a power of two) fit into the list in part (a)?

4. Find exact closed-form solutions to the following recurrences.

   (a) $g(0) = 0$, $g(1) = 3$, $g(n) = g(n-1) + 2g(n-2)$ for $n \geq 2$.

   (b) $T(n) = 3T(n/3) + 3n$, $T(1) = 0$, n is a power of 3.

   (c) $F(n) = 4F(n/4) + n^2$, $F(1) = 1$, n is a power of 4.

5. The Master Theorem states that the solution to the recurrence

$$T(n) = aT(n/b) + g(n),$$

where a and b are constants, has asymptotic behaviour

$$T(n) = \begin{cases} \Theta(n^{\log_b a}), & \text{if } g(n) = O(n^c) \text{ for every } c < \log_b a \\ \Theta(g(n)), & \text{if } g(n) = \Omega(n^c) \text{ for every } c > \log_b a. \end{cases}$$

Use Theorem 5 from Lecture 8 (the Akra-Bazzi Theorem) to prove the Master Theorem. Assume that $|dg(x)/dx| = O(x^C)$ for some constant C.

6. You want to move the Towers of Hanoi, but now you have four poles instead of three. The rules are the same: n disks are initially stacked up from largest at the bottom to smallest on top on the leftmost pole. The objective is to move them to the rightmost pole one by one so that at no point does a larger disk cover a smaller one.

   Consider the following strategy: If $n \leq 10$, ignore one of the poles and apply the solution from class for three poles. If $n > 10$, recursively move the top $n - 10$ disks to the second pole, stack up the bottom 10 disks onto the last pole using the other three poles only, and then recursively move the $n - 10$ remaining disks from the second pole to the last pole.

   Let $T(n)$ be the number of steps that it takes to move the whole stack of n disks.

   (a) Write a recurrence for $T(n)$. Explain why your recurrence is correct.

   (b) Show that the recurrence from part (a) satisfies $T(n) = O(2^{n/10})$.

   (c) **(Extra credit)** Can you come up with a different strategy in which $2^{O(\sqrt{n})}$ moves are sufficient?