Each of the questions is worth 10 points. Please turn in solutions to *four* questions of your choice. Write your name, your student ID, and your TA's name on the solution sheet.

Please write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions and list your collaborators on your solution sheet. Copying someone else's solution will be considered plagiarism and may result in failing the whole course.

Questions

1. Find the mistakes in the following "proofs".

   (a) **Theorem:** $2 = 1$.

   Proof. Let a, b be integers such that $a = b$. Multiplying both sides by a, we obtain $a^2 = ab$. After adding $a^2 - 2ab$ to both sides, we get that $2a^2 - 2ab = a^2 - ab$. We can rewrite this equality as $2(a^2 - ab) = a^2 - ab$. After cancelling $a^2 - ab$, we conclude that $2 = 1$. □

   (b) **Theorem:** The iPhone 5 is better than the iPhone 6.

   Proof. The iPhone 5 is better than nothing, and nothing is better than the iPhone 6. Therefore, the iPhone 5 is better than the iPhone 6. □

   (c) **Theorem:** Every collection of 8 people includes a group of 4 friends or a group of 4 strangers.

   Proof. Let a denote one of the six people. The proof is by case analysis. We consider two cases:

   - **Case 1:** a is friends with at least 4 other people in the collection.
   - **Case 2:** a is a stranger to at least 4 other people in the collection.

   One of these two cases must hold. Let's discuss Case 1. If all the people who are friends with a are strangers among themselves, this is a group of 4 strangers. Otherwise, at least 3 of them are mutual friends, and together with a they form a group of 4 friends.

   Now let's do Case 2. If all the people who are strangers to a are friends among themselves, this is a group of 4 friends. Otherwise, at least 3 of them are mutual strangers, and together with a they form a group of 4 strangers. □

2. Verify the following proofs. If the proof is invalid, explain the problem with it.

   (a) For any integer n, $n^3 - n$ is divisible by 3.

   Proof. We will prove this proposition by case analysis.
   Case 1: if $n = 3k + 1$ for some k, then $n^3 - n = 27k^3 + 27k^2 + 6k$, which is divisible by 3.
   Case 2: if $n = 3k + 2$ for some k, then $n^3 - n = 27k^3 + 54k^2 + 33k + 6$, which is also divisible by 3.
   It follows that for any integer n, $n^3 - n$ is divisible by 3. □

(b) **Theorem:** For every integer n, n is odd if and only if n^2 is odd.

Proof. First, we prove that if n is odd then n^2 is also odd. If n is odd, then $n = 2k+1$ for some integer k. Hence, $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. This is an even number plus one, which is an odd number.

Then, we prove that if n^2 is even, then n is also even. By Theorem 8 from the Lecture 2 notes, this is true. We conclude that the proposition holds for all n. □

(c) **Theorem:** Every integer is rational.

Proof. We prove this by contradiction. Assume, for contradiction, that every integer is irrational. Then, in particular, 2 is irrational. But we can write 2 as $6/3$, which is a ratio of integers and therefore rational. Contradiction. □

3. Which of the following propositions is true? If a proposition P is true, prove it. If it is false, disprove it (that is, prove the proposition NOT P).

   (a) Every collection of 7 people includes a group of 3 friends or a group of 3 strangers.

   (b) Every collection of 5 people includes a group of 3 friends or a group of 3 strangers.

   (c) (**Extra Credit**) Every collection of 9 people includes a group of 4 friends or a group of 3 strangers.

4. Prove the following theorems.

   (a) **Theorem:** $\sqrt{3}$ is irrational.

   (b) **Theorem:** $\log_{\sqrt{2}} 3$ is irrational.

5. Prove that for all integers $n > 1$, $(n^2 + 1)/(n + 1)$ is not an integer.

6. Prove the following theorem. Let m and n be positive integers with no common factor greater than 1. Tile the plane with 1×1 squares whose vertices are points with integer coordinates and draw a line segment s from the point with coordinates $(0, 0)$ to the point with coordinates (m, n).

   Theorem: The line segment s passes through exactly $m + n - 1$ of the squares.

   Here is an example with $m = 3$, $n = 2$. The line segment from $(0, 0)$ to $(3, 2)$ passes through the 4 gray squares.