Practice Final 1

1. Write the proposition "There is at most one ball in every urn" using logical connectives and quantifiers. Use the symbols b_1, b_2 for balls, u_1, u_2 for urns and $IN(b, u)$ for "ball b is in urn u".

   Solution: $\forall u, b_1, b_2 \colon IN(b_1, u)$ AND $IN(b_2, u) \longrightarrow b_1 = b_2$. Any two balls in any given urn must in fact be the same ball.

2. A *cut-edge* in a connected graph is an edge e such that if e was removed, the graph would no longer be connected. Show that any connected graph in which all vertices have even degree does not have a cut-edge.

   Solution: We prove this is impossible by contradiction. Suppose there exists a connected graph G with exactly one cut-edge $e = \{u, v\}$ in which all vertices have even degree. Let G' be the graph obtained by removing the edge e from G. Then u and v must belong to different connected components C and C' of G' (for otherwise removing e from G would not disconnect it.) All the vertices of C except for u have even degree, so C has exactly one vertex of odd degree. Therefore the sum of the degrees of the vertices in C is odd. This is impossible: In lecture 5 we showed that the sum of the degrees of all vertices in a graph (and therefore in all of its connected components) must be even.

3. The graph G_1 consists of a single vertex. For $n \geq 1$, the graph G_{n+1} consists of two disjoint copies of G_n and a matching between the vertices of the two copies. How many edges does G_n have?

   Solution: The graph G_{n+1} has twice as many vertices as G_n for all n, so G_n must have 2^{n-1} vertices. Let $f(n)$ be the number of edges of G_n. Since G_{n+1} contains twice the edges of G_n plus 2^{n-1} edges from the matching, f satisfies the recurrence

   $$f(n+1) = 2f(n) + 2^{n-1}$$

   with base case $f(1) = 0$. We guess a solution for $f(n)$ by iterating this formula:

   $$\begin{aligned} f(n) &= 2f(n-1) + 2^{n-2} \\ &= 2(2f(n-2) + 2^{n-3}) + 2^{n-2} = 2^2 \cdot f(n-2) + 2 \cdot 2^{n-2} \\ &= 2^2 \cdot (2f(n-3) + 2^{n-4}) + 2 \cdot 2^{n-2} = 2^3 \cdot f(n-3) + 3 \cdot 2^{n-2}. \end{aligned}$$

   This suggests the guess $f(n) = 2^{n-1} \cdot f(1) + (n-1) \cdot 2^{n-2} = (n-1) \cdot 2^{n-2}$.

   We now prove that $f(n) = (n-1) \cdot 2^{n-2}$ by induction on n. When $n = 1$, $f(1) = 0$ and $(n-1) \cdot 2^{n-2} = 0$. Now assume $f(n) = (n-1) \cdot 2^{n-2}$ for some $n \geq 1$. Then

   $$f(n+1) = 2f(n) + 2^{n-1} = 2 \cdot (n-1) \cdot 2^{n-2} + 2^{n-1} = n \cdot 2^{n-1}$$

   so the formula must be correct for all n.

4. Let $f(n) = 1 + 1/3 + 1/5 + \cdots + 1/(2n-1)$. Show that f is $\Theta(\log n)$.

   Solution: On the one hand, $f(n) \le 1 + 1/2 + 1/3 + \cdots + 1/(2n) = H(2n)$, where $H(n)$ is the n-th harmonic number from Lecture 7. There we showed that $H(n)$ is $O(\log n)$, so $H(2n)$ and therefore $f(n)$ is also $O(\log n)$.

   On the other hand, $f(n) \ge 1/2 + 1/4 + 1/6 + \cdots + 1/(2n) = \frac{1}{2}H(n)$. In Lecture 7 we also showed that $H(n)$ is $\Omega(\log n)$, so $f(n)$ is also $\Omega(\log n)$.

5. You drop 30 balls into 7 urns. Some of the balls are red and some are blue. Show that at least three balls of the same colour land in the same urn.

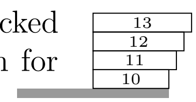
   Solution: Let $f\colon \{1,\dots,30\} \to \{1,\dots 7\} \times \{\text{red}, \text{blue}\}$ be the function that assigns each ball to its urn and its colour. The domain of f has size 30 and its range has size $7 \cdot 2 = 14$. Since $30 > 2 \cdot 14$, by the generalized pigeonhole principle there exist 3 balls that are assigned to the same urn and have the same colour.

6. You are dealt 5 random cards from a 52-card deck. What is the probability that the largest face value is a 9? (The face values from smallest to largest are 2 3 4 5 6 7 8 9 10 J Q K A.)

   Solution: Let A be the set of hands in which the largest face value is a 9. Let B be the set of hands in which all face vales are *at most* 9 and C be the set of hands in which all face values are at most 8, respectively. Then B is the disjoint union of A and C, so $|B| = |A| + |C|$.

   The set B consists of all 5-card hands out of the $8 \cdot 4 = 32$ cards with face values 2 up to 9 in all four suits, so $|B| = \binom{32}{5}$. Similarly, $|C| = \binom{28}{5}$ and so $|A| = \binom{32}{5} - \binom{28}{5}$. Assuming equally likely outcomes, the probability of getting a hand in the set A is then $\left(\binom{32}{5} - \binom{28}{5}\right)/\binom{52}{5} = (201,376 - 98,280)/2,598,960 \approx 0.0397$. *(You may omit the calculation.)*

7. You have overhang blocks 10, 11, up to n units long, one of each kind. They are stacked over the table from smallest to largest so that their left edges align. (See diagram for $n = 13$). Show that the configuration is not stable when n is sufficiently large.



   Solution: We assume all blocks have the same weight. If instead a block's weight is proportional its length the calculation is a bit more complicated but the conclusion is similar.

   The center of mass of all the blocks, measured from the left edge of the blocks, is at position

$$P(n) = \frac{1}{2n} \cdot (10 + 11 + \cdots + n) = \frac{1}{2n}\big((1 + \cdots + n) - (1 + \cdots + 9)\big) = \frac{1}{2n}\left(\frac{n(n+1)}{2} - \frac{9 \cdot 10}{2}\right) = \Omega(n)$$

   so when n is sufficiently large, $P(n) > 10$, the center of mass falls to the right of the edge of the table, and the configuration is not stable.

Practice Final 2

1. Show that for every integer n, if $n^3 + n$ is divisible by 3 then $2n^3 + 1$ is *not* divisible by 3.

   Solution: We can prove this proposition by cases depending on the residue of $n^3 + n$ modulo 3. If $n \equiv 0 \bmod 3$ then $n^3 + n$ is divisible by 3, while $2n^3 + 1 \equiv 1 \bmod 3$, so $2n^3 + 1$ is not divisible by 3, so the proposition holds. If $n \equiv 1 \bmod 3$ then $n^3 + n \equiv 2 \bmod 3$, so $n^3 + n$ is not divisible by 3 and the proposition holds again. If $n \equiv 2 \bmod 3$, then $n^3 + n \equiv 2 \bmod 3$ and $n^3 + n$ is not divisible by 3 again.

2. The vertices of graph G are the integers from 1 to 20. The edges of G are the pairs $\{x, y\}$ such that $\gcd(x, y) > 1$. How many connected components does G have?

   Solution: Six. The vertex 1 clearly forms a connected components. So does each of the vertices 11, 13, 17, and 19: These are all prime numbers that are greater than 10, so their gcd with any number between 1 and 20 equals one. We now argue that all the remaining vertices are in a single connected component. These vertices are the multiples of 2, 3, 5, and 7, so each remaining vertex is connected to 2, 3, 5, or 7. It remains to show that 2, 3, 5, and 7 are in the same connected component. This is true because G contains the paths $(2, 6, 3)$, $(2, 10, 5)$, and $(2, 14, 7)$.

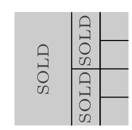
3. What is $1 + (1 + 2) + (1 + 2 + 3) + \cdots + (1 + 2 + 3 + \cdots + 1000)$?

   Solution: The sum of the first k integers is $k(k + 1)/2 = \frac{1}{2}k^2 + \frac{1}{2}k$, so

$$\begin{aligned} 1 + (1 + 2) + \cdots + (1 + 2 + 3 + \cdots + n) &= \tfrac{1}{2}(1^2 + 2^2 + \cdots + n^2) + \tfrac{1}{2}(1 + 2 + \cdots + n) \\ &= \tfrac{1}{2}(\tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n) + \tfrac{1}{2}(\tfrac{1}{2}n^2 + \tfrac{1}{2}n) \\ &= \tfrac{1}{6}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{3}n \end{aligned}$$

   using the formulas for the sum of the first n squares of integers and the sum of the first n integers, respectively. (Notice that this expression gives the correct answer for $n = 0$, 1, and 2.) Plugging in $n = 1000$ we obtain the answer $\frac{1}{6} \cdot 10^9 + \frac{1}{2} \cdot 10^6 + \frac{1}{3} \cdot 10^3 = 167,167,100$.

4. An $n \times n$ plot of land (n is a power of two) is split in two equal parts by a North-South fence. The Western half is sold and the Eastern half is split in two equal parts by an West-East fence. The same procedure is applied to the remaining $(n/2) \times (n/2)$ plots until 1×1 plots are obtained (see $n = 4$ example). How many units of fence are used?



   Solution: The amount $T(n)$ of fence used satisfies the recurrence $T(n) = 2T(n/2) + 3n/2$ for $n > 1$, with $T(1) = 0$. We can unwind the recurrence as follows:

$$\begin{aligned} T(n) &= 2T(n/2) + 3/2 \cdot n \\ &= 2(2T(n/2^2) + 3/2 \cdot n/2) + 3/2 \cdot n = 2^2T(n/2^2) + 3/2 \cdot 2n \\ &= 2^2(2T(n/2^3) + 3/2 \cdot n/2^2) + 3/2 \cdot 2n = 2^3T(n/2^3) + 3/2 \cdot 3n \end{aligned}$$

   After $\log n$ steps we expect to obtain $T(n) = n \cdot T(1) + \frac{3}{2}n \log n = \frac{3}{2}n \log n$. We confirm the correctness of this guess by induction. For the base case $n = 1$, $T(1) = 0$ as desired. For the inductive step we assume $T(k) = \frac{3}{2}k \log k$ for all $k < n$ that are powers of two. Then

$$T(n) = 2T(n/2) + 3n/2 = 2 \cdot \frac{3}{2} \cdot \frac{n}{2} \log(n/2) + \frac{3n}{2} = \frac{3n}{2} \cdot (\log n - 1) + \frac{3n}{2} = \frac{3}{2} \cdot n \log n$$

   when n is a power of two, concluding the inductive step.

5. A department has 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

   Solution: There are $\binom{10}{3}$ ways to choose the three men in the committee and $\binom{15}{3}$ ways to choose the three women in it, so the number of possible committees is $\binom{10}{3} \cdot \binom{15}{3} = 54,600$.

6. A password is made of the digits `0`, `1`, $\ldots$, `9` and the special symbols `*` and `#`. The password must be 4-6 symbols long and contain at least one special symbol. How many passwords are there?

   Solution: Let P be the set of possible passwords. Then P is the disjoint union of P_4, P_5, and P_6, where P_ℓ is the set of possible passwords of length ℓ. The set of all ℓ-symbol strings is the disjoint union of P_ℓ and the set N_ℓ of ℓ-symbol strings that contain no special symbol, so P_ℓ has $12^\ell - 10^\ell$ elements. By the sum rule,

   $$|P| = |P_4| + |P_5| + |P_6| = (12^4 - 10^4) + (12^5 - 10^5) + (12^6 - 10^6) = 2,145,552.$$

7. Show that every set of 10 integers, each of them between 0 and 25, contains two distinct subsets S, T *of the same size* such that the sum of the numbers in S equals the sum of the numbers in T.

   Solution: Let X be the set of all possible 5-element subsets of the 10-integer set and $f\colon X \to \{0, 1, \ldots, 125\}$ be the function $f(\{a, b, c, d, e\}) = a + b + c + d + e$. Since X has size $\binom{10}{5} = 252$, which is greater than 126, by the pigeonhole principle there must exist two subsets S and T such that $f(S) = f(T)$.

Practice Final 3

1. Show that if x is irrational and y is any real number then at least one of $x + y$ and $x - y$ must be irrational.

   Solution: We prove the contrapositive. Assume $x + y$ and $x - y$ are both rational. Then so is $\frac{1}{2}(x + y) - \frac{1}{2}(x - y) = x$.

2. A box contains 100 black balls and 99 white balls. In each step Alice takes out two balls of the same colour and puts in one ball of the opposite colour. Can Alice ~~empty the box~~ be left with exactly one ball of each colour in the box?

   Solution: No. We show that the predicate "3 divides $b - w - 1$" is an invariant of the underlying state machine, where b and w is the number of black and white balls respectively. The invariant holds initially. We now argue that it is preserved by the transitions, so assume 3 divides $b - w + 1$ before a given transition. There are two possibilities after the transition: Either the box contains $b-2$ black and $w+1$ white balls, in which case $(b-2)-(w+1)-1 = (b-w-1)-3$ is a multiple of 3, or the box contains $b+1$ black and $w-2$ white balls, in which case $(b+1)-(w-2)-1 = (b-w-1)+3$ is also a multiple of 3. Since 3 does not divide $1 - 1 - 1 = -1$, the state in which there is exactly one ball of each colour cannot be reached.

3. In a group of 15 people, is it possible for each person to have exactly 3 friends? (If Alice is a friend of Bob we assume Bob is also a friend of Alice.)

   Solution: No. Suppose for contradiction this was possible. Then the sum of degrees in the friendship graph would have been $15 \cdot 3 = 45$. But the sum of the degrees equals twice the number of edges, which is an even number, contradicting the fact that 45 is odd.

4. Sort these three functions in increasing order of growth: $\sqrt{n} \cdot \log n$, $n/\sqrt{\log n}$, $\sqrt{n \cdot \log n}$. For your sorted list f, g, h show that f is $o(g)$ and g is $o(h)$.

Solution: $\sqrt{n \log n}$ is $o(\sqrt{n} \log n)$ because the ratio $\sqrt{n \log n}/\sqrt{n} \log n$ equals $1/\sqrt{\log n}$, which eventually becomes and stays smaller than any given constant. $\sqrt{n} \log n$ is $o(n/\sqrt{\log n})$ because the ratio $\sqrt{n} \log n/(n/\sqrt{\log n})$ equals $(\log n)^{3/2}/n^{1/2}$. In Lecture 7 we showed that $(\log n)^a$ is $o(n^b)$ for any constants $a, b > 0$, so this ratio becomes and stays smaller than any constant when n is sufficiently large.

5. The vertices of graph H are the 20 integers from -10 to 10 except 0. The edges of H are the pairs $\{x, y\}$ such that $x = -y$ or $|y - x| = 1$. How many perfect matchings does H have?

   Solution: Let $f(n)$ denote the number of matching of the analogous graph H_n with $2n$ vertices in which the integers -10 and 10 are replaced by $-n$ and n. There are two possible ways in which vertex n can be matched: Either it is matched to $-n$, in which case the remaining vertices to be matched induce the graph H_{n-1}, or it is matched to $n - 1$, in which case $-n$ must also be matched to $-(n+1)$ and the remaining vertices to be matched induce the graph H_{n-2}. Therefore the number of matchings $f(n)$ satisfies the recurrence $f(n) = f(n-1) + f(n-2)$ for all $n \geq 2$. By inspection we have that $f(0) = 1$ and $f(1) = 1$. This is exactly the same recurrence we had in Lecture 7 and we can calculate the following values for $f(n)$ when $n \leq 10$:

n	0	1	2	3	4	5	6	7	8	9	10
$f(n)$	1	1	2	3	5	8	13	21	34	55	89

   so $f(10) = 89$.

6. How many length 5 passwords are there that contain at least one digit ($\texttt{0}, \texttt{1}, \ldots, \texttt{9}$), at least one `*`, and at least one `#`? No other symbols are allowed.

   Solution: Let A, B, and C denote the sets of length 5 strings (with the given symbols) that contain no digits, no `*`, and no `#`, respectively. The set of passwords is the complement of the set $A \cup B \cup C$, so there is a total of $12^5 - |A \cup B \cup C|$ passwords. By inclusion-exclusion,

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 2^5 + 11^5 + 11^5 - 1^5 - 1^5 - 10^5 + 0 \end{aligned}$$

   because the set A is the product set $\{*, \#\}^5$, the set $B \cap C$ is the product set $\{\texttt{0}, \texttt{1}, \ldots, \texttt{9}\}^5$, and so on. So the number of passwords is

$$12^5 - 2^5 - 2 \cdot 11^5 + 2 + 10^5 = 26,700.$$

7. Prove that every tree can have at most one perfect matching.

   Solution: The proof is by strong induction on the number of vertices. If a tree has one vertex then it has no perfect matching so the proposition holds. Now assume it is true for all trees with fewer than n vertices and consider any tree T with n vertices. T must have a vertex v of degree one. This vertex v can be matched in at most one way to its unique neighbor w. We now argue that there exists at most one matching that covers all remaining vertices. The graph G obtained by removing v and w from T with all their incident edges is a forest. By the inductive assumption, each connected component of G can have at most one perfect matching, so G itself, and therefore T also, can have at most one perfect matching.

An alternative proof is to argue the contrapositive: A union of any two distinct perfect matchings Ξ_0 and Ξ_1 on the same set of vertices must contain a cycle, so Ξ_0 and Ξ_1 cannot both be perfect matchings of a tree. (Distinct does not mean *disjoint*: Ξ_1 and Ξ_2 may share some edges.) To prove this, let v_1 be any vertex that is matched differently in Ξ_0 and Ξ_1 and v_0 be its match in Ξ_0. Consider the sequence of vertices $v_0, v_1, v_2, v_3, \cdots$ where v_2 is v_1's match in Ξ_0, v_3 is v_2's match in Ξ_1, v_4 is v_3's match in Ξ_0, and so on; the matchings alternate as vertices are added. At some point a repeated vertex $v_j = v_i$ with $j > i$ must appear in the sequence. We now argue that $j \neq i + 2$, so $v_i, v_{i+1}, \ldots, v_{j-1}$ is the desired cycle. In fact, for every $i \geq 0$, v_i and v_{i+2} must be distinct. We can prove this by induction on i: this is true when $i = 0$ by the assumption on v_1, and given that v_i and v_{i+2} are distinct, their matches v_{i+1} and v_{i+3} must also be distinct.