ENGG 2440A: Discrete Mathematics for Engineers | Homework 4
The Chinese University of Hong Kong, Fall 2017 | due Thursday 9 November

Questions 1 to 6 are worth 10 points each. Please turn in solutions to *four* questions of your choice. Write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions and give credit to your collaborators on your solution sheet. Copying someone's solution or pasting material you found online without reference will be considered plagiarism and may result in failing the whole course.

Questions

1. Are the following propositions about graphs true or false? Justify your answer by giving a proof or a relevant example.

   (a) Assume G is connected. Let G' be the graph obtained by removing an edge e from G. G' is connected if and only if e lies on a cycle in G.

   (b) Assume G is connected. Let G' be the graph obtained by removing a vertex v from G together with its incident edges. G' is connected if and only if v lies on a cycle in G.

   (c) A *leaf* is a vertex of degree 1. The number of leaves in a forest is larger than the number of its connected components.

   (d) If every vertex belongs to at least one closed walk of odd length then there are at least as many edges as there are vertices.

2. Find exact closed form expressions for the following sums. Explain how you discovered the expression and prove that it is correct.

   (a) $(4-3)/5 + (4^2 - 3^2)/5^2 + \cdots + (4^n - 3^n)/5^n$.

   (b) $1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1) \cdot n$.

   (c) $-1^2 + 2^2 - 3^2 + 4^2 - \cdots - (n-1)^2 + n^2$, where n is even.

   (d) $1 \cdot 2^1 + 2 \cdot 2^2 + \cdots + n \cdot 2^n$. (**Hint:** Take derivatives in the geometric sum formula.)

3. Show the following inequalities.

   (a) $\frac{3}{4}n^{4/3} - \sqrt[3]{n} \le 1 + \sqrt[3]{2} + \cdots + \sqrt[3]{n-1} \le \frac{3}{4}n^{4/3}$.

   (b) $1.46 \le 1 + 1/2^2 + 1/3^2 + \cdots \le 1.67$. (**Hint:** Split the summation into two parts.)

   (c) $1 - 1/2 + 1/3 - 1/4 + \cdots + 1/(2n-1) - 1/2n \le 1$.

4. Bob's grocery shop orders fresh milk from a supplier at a price of 10 dollars per litre. His objective is to sell it at maximum profit.

   (a) Bob's research shows that on day the milk arrives, he can sell $100 - x$ litres of milk at x dollars per litre. How should Bob set the price to maximize his profit on this day?

   (b) Bob estimates that his sales volume will drop by 20% per day, assuming that the price stays the same. How much can he profit over the lifetime of his supply? How much milk should he order? Can he increase his profit by changing prices on different days?

   (c) Bob's consultant Alice claims that his model from part (b) is incorrect. Instead, if Bob discounts the price of milk by 20% per day, then his sales volume remains the same. How do the answers from part (b) change in Alice's model?

5. Consider the following directed acyclic graph G. The vertices of G are the integers from 1 to 20. The pair (u, v) is an edge of G if (and only if) u divides v and $u \neq v$.

   (a) Show that G has a parallel schedule of duration 4 but not one of duration 3.

   (b) Show that G has an antichain of size 10.

   (c) Show that there exist 10 vertex-disjoint paths in G that cover all 20 vertices.

   (d) Show that G has no antichain of size 11.

   (e) **(Extra Credit)** If the vertices of G were the integers from 1 to n what would be the longest path and the largest antichain in G? Provide a proof.

6. This question concerns spanning trees of the hypercube H_n which was defined in question 6 of Homework 3.

   (a) Give a spanning tree for H_3 that contains no path of length 6.

   (b) Show that H_n has a spanning tree that contains no path of length $2n$. (**Hint:** Use induction. You may need to prove a stronger proposition.)

   (c) Show that for all $n \geq 2$, every spanning tree of H_n contains a path of length $n + 1$.

   (d) **(Extra credit)** Does every spanning tree of H_n contain a path of length $2n - 1$?