Questions 1 to 6 are worth 10 points each. Please turn in solutions to *four* questions of your choice. Write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions and give credit to your collaborators on your solution sheet. Copying someone's solution or pasting material you found online without reference will be considered plagiarism and may result in failing the whole course.

Questions

1. For each pair of propositions, say whether they are logically equivalent. Justify your answer. For propositions stated in plain English, formulate them using logical symbols first.

   (a) P OR Q
   (P XOR Q) OR Q

   (b) (P IFF Q) AND (Q IFF R)
   P IFF R

   (c) Not all three balls are of the same colour. (Colours are black or white.)
   Among the three balls, there is exactly one white ball, or there is exactly one black ball.

   (d) If Alice or Bob goes to the concert, then Charlie won't go.
   Alice, Bob, and Charlie won't all go to the concert.

   (e) **Extra credit.** $P_1 \longrightarrow (P_2 \longrightarrow (P_3 \longrightarrow (P_4 \longrightarrow (P_5 \longrightarrow P_6))))$
   (NOT (P_1 AND P_2 AND P_3 AND P_4 AND P_5)) OR P_6

2. The following propositions are about a group of inmates at Cape Collinson Correctional Institution. Some of the inmates are planning an escape, and some of the inmates are actually "plants" – policemen in disguise as inmates. $Plant(x)$ means that person x is a plant. $Knows(x, y)$ means that person x knows person y's escape plans (whether they plan an escape or not). Translate the following propositions into plain English.

   (a) $\forall x, y : (Plant(x)$ AND $Plant(y)) \longrightarrow Knows(x, y)$

   (b) $\exists x, y : (x \neq y)$ AND $(\forall z : Plant(z) \longrightarrow (Knows(z, x)$ AND $Knows(z, y)))$

   (c) $\forall x : (Plant(x)$ AND $\forall y : Knows(x, y)) \longrightarrow (x = \text{Alice})$

   (d) $\forall z :$ NOT $Knows(z, z)$

   (e) $\exists x \forall y :$ (NOT $Plant(x)$) AND $Knows(x, y)$

3. Translate the following English sentences into propositions using quantifiers and logical symbols. You may use $N(x, y)$ for "MTR stations x and y neighbour each other" and $C(x, y)$ for "the (passenger) capacity of station x is strictly larger than that of station y".

   (a) There exist neighbouring stations with different capacities.

   (b) Any two neighbouring stations have the same capacity.

   (c) No station neighbours all other stations.

   (d) The station with the lowest capacity neighbours the station with the highest capacity.

   (e) There are at least three stations with the lowest capacity.

4. Say which among the following pairs of propositions with quantifiers are logically equivalent. Justify your answer.
To say that two propositions are equivalent, describe their common meaning in English. To argue that two propositions are not equivalent, describe a possible world in which one of them is true and the other false. (For example, in a world where Alice is *R*ich but Bob is not, the proposition $\exists x\colon R(x)$ is true but $\forall x\colon R(x)$ is false, so the two cannot be equivalent.)

   (a) $(\exists x : P(x))$ AND $(\exists y : Q(y))$
   $\exists x : P(x)$ AND $Q(x)$

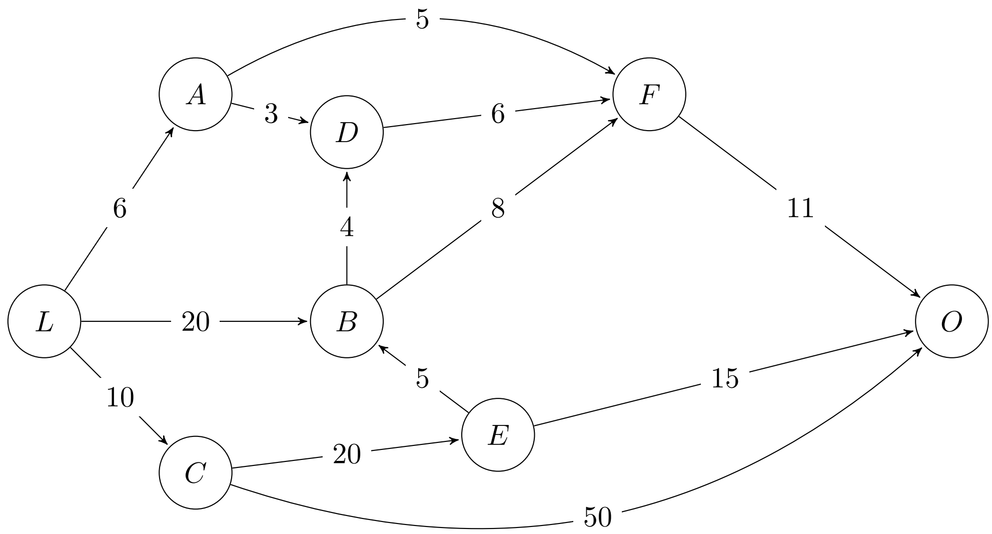
   (b) $\forall x : P(x)$ XOR $Q(x)$
   $\exists x : P(x)$ AND NOT $Q(x)$

   (c) $(\forall x : P(x))$ AND $(\forall y : Q(y))$
   $\forall x, y : (P(x)$ AND $Q(y))$

   (d) $\exists x : P(x) \longrightarrow Q(x)$
   $\forall x : Q(x)$ OR NOT $P(x)$

   (e) $\exists x \forall y, z : P(x, y, z)$
   $\forall z \exists x \forall y : P(x, y, z)$

5. You are organizing galactic traffic. Below is a map of planets and allowed spacecraft routes, each labelled by the number of spacecraft that are allowed to use the route. Your goal is to move as many spacecrafts as possible from "Lagrange City" (L) to "Oortville" (O).



   (a) How should you route the spacecraft so that as many of them reach Oortville as possible?

   (b) Explain convincingly why no additional spacecraft can make it to Oortville.

   (c) You are considering upgrading some of the routes to wormholes. An unlimited number of spacecraft are able to move through a wormhole (i.e. they have infinite capacity). What is the fewest number of upgrades you need so that any number of spacecraft can make the journey?

   (d) **Extra credit.** How can you use wormhole upgrades so as to maximise the number of spacecraft that can make the journey, but so that it remains impossible for infinitely many spacecraft to get to Oortville?

6. Express the following predicates about numbers (non-negative integers) using quantified formulas and the symbols 0, 1, =, +, ×, and E. The first five symbols have their usual meaning and $m\mathrm{E}n$ stands for m^n. For example, "$m = \sqrt{n} + 1$" can be expressed as $\exists r : m = r + 1$ AND $n = r \times r$.

   (a) i. $m \leq n$ and ii. $m < n$.

   (b) $m \bmod q = r$ meaning "r is the remainder when m is divided by q."

   A (binary) string is a finite sequence of `0`s and `1`s. We will represent the string x by the number whose binary expansion is `1`x. For instance, string `0` is represented by number 2, string `110` is represented by number 14, and the empty string is represented by number 1.

   With this convention, predicates about strings can be expressed as predicates about numbers. For example, the predicate "the last bit of (string) x is `1`" is expressed by "$x \bmod 2 = 1$." Express the following predicates:

   (c) $\mathrm{bit}(x, m)$ meaning "the m-th bit of string x is `1`".

   (d) $\mathrm{len}(x, m)$ meaning "string x has length m".

   (e) $z = x \circ y$ meaning "string z is the concatenation of strings x and y".
   (Concatenation means z is obtained by typing first x then y, e.g., `10011` = `10` ∘ `011`.)

   (f) **(Extra credit)** $m = 1 \times 2 \times \cdots \times n$.
   (**Hint:** Although this question is about *numbers*, strings may come in handy.)