1. Let G be a graph with 10 vertices and 9 edges. Is it true that G must be a tree? Justify your answer.

   Solution: No. For example, the graph G that consists of one cycle on 9 vertices and one isolated vertex (two connected components) has 10 vertices and 9 edges but is not a tree.

2. Alice places two pebbles at the opposite corners of an 8 by 8 chessboard. At each step, she can

   - put a new pebble in an empty square, if *exactly one* of its neighbors contains a pebble, or
   - remove a pebble from a square, if *at least one* of its neighbors contains a pebble.

   Neighbors are squares that share a common side. Can the board ever have a single pebble on it?

   Solution: No. Let G be the graph whose vertices are the pebbles and edges are pebbles that are neighbors. The predicate "G has at least two connected components" is an invariant of this state machine. Initially G has two connected components so the invariant holds. We show that the number of connected components cannot decrease after a transition. For the first type of move it remains the same because the new pebble extends an existing component without affecting the others. For the second type of move, the component that the removed pebble belongs to breaks up into one or more components, while the other components are unaffected, so the number of components cannot decrease. A board with a single pebble has one connected component so this state can never be reached.

3. The set S_n consists of all length-n strings with symbols $\{\texttt{A}, \texttt{B}, \texttt{C}\}$ in which every `B` is immediately followed by a `C` (e.g., `BCAC` is in S_4 but `ACAB` is not). Find the value of a for which $|S_n|$ is $\Theta(a^n)$.

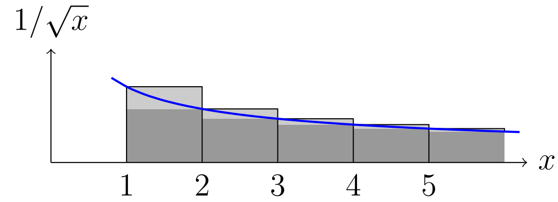
   Solution: Let $f(n) = |S_n|$. The set S_n is the disjoint union of $\{\texttt{A}\} \times S_{n-1}$, $\{\texttt{C}\} \times S_{n-1}$ and $\{\texttt{BC}\} \times S_{n-2}$, so f satisfies the recurrence $f(n) = 2f(n-1) + f(n-2)$ for all $n \geq 2$. Solutions of the form $f(n) = a^n$ must therefore satisfy $a^2 - 2a - 1 = 0$. There are two such solutions: $a_1 = 1 + \sqrt{2}$ and $a_2 = 1 - \sqrt{2}$. The solution of the recurrence must then be of the form $f(n) = c_1(1+\sqrt{2})^n + c_2(1-\sqrt{2})^n$, where c_1 and c_2 should be chosen to satisfy the initial conditions. Regardless of the values of c_1 and c_2, $f(n)$ is $\Theta((1+\sqrt{2})^n)$, so $a = 1 + \sqrt{2}$.

4. What is the largest integer n for which

   $$n \leq 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{9999}}?$$

   Solution: Let S denote the sum on the right. The area under the first 9999 rectangles is at least as large as the area under the curve, so

   $$S \geq \int_1^{10000} \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_1^{10000} = 200 - 2 = 198.$$



If the area L under the light rectangles is removed from S then the dark rectangles fit under the curve, so $S - L \leq 198$. The light rectangles stack up to a rectangle of width 1 and height less than 1, so $L < 1$. Therefore $198 \leq S < 199$ and $n = 198$.

5. At a party, seven people check in their hats. In how many ways can they be returned so that *exactly one* person receives their own hat? Show your calculations.

   Solution: Let A_i be the set of permutations of hats in which person i receives their own hat and none of the others do. We are interested in the size of $A = A_1 \cup A_2 \cup \cdots \cup A_7$. Since the sets A_i are disjoint, the size of this set is $|A_1| + |A_2| + \cdots + |A_7|$.

   To calculate the size of A_i, ignore person i and their hat; what remains are permutations of the others' six hats in which no person gets their own hat back. By the sum rule, this number is 6! minus the number of permutations in which at least one of six people gets their hat back. By the calculation in Lecture 10,

$$|A_i| = 6! - 6!\left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!}\right) = \frac{6!}{2!} - \frac{6!}{3!} + \frac{6!}{4!} - \frac{6!}{5!} + \frac{6!}{6!} = 265$$

   so $|A| = 7 \cdot 265 = 1,855$.

6. Show that no matter how you place 17 pieces on an 8 by 8 chessboard, at least two pieces must occupy squares that share a common side or a common corner.

   Solution: Divide up the chessboard evenly into sixteen 2×2 blocks. Any two squares within a block share a common side or a common corner. By the pigeonhole principle, some two pieces must be placed in the same block, so they will share a common side or a common corner.

7. Suppose that all arrangements of n plus signs and n minus signs in a row are equally likely. Give a formula for the probability that no two minus signs are adjacent to each other. Specify the relevant sample space and event. Show your calculations.

   Solution: The sample space S consists of all arrangements of n plus and n minus signs in a line. We model signs of the same type as indistinguishable, so $|S| = \binom{2n}{n}$. The event E of interest consists of those arrangements in which no minus signs are consecutive. The n minus signs must fall in the $n+1$ gaps between plus signs (including the leading and final gap), so $|E| = \binom{n+1}{n} = n+1$. By the formula for equally likely outcomes, $\Pr[E] = |E|/|S| = (n+1)/\binom{2n}{n}$.