ENGG 2430 / ESTR 2004: Probability and Statistics

Spring 2019

5. Conditioning and Independence

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Conditional PMF

Let X be a random variable and A be an event.

The conditional PMF of X given A is

$$P(X = x \mid A) = \frac{P(X = x \text{ and } A)}{P(A)}$$

What is the PMF of a 6-sided die roll given that the $\underbrace{\text{outcome is even}}_{\rightarrow E}$?

$$p_X(x) = \frac{1}{6} \quad \text{FOR} \quad x = 1, 2, 3, 4, 5, 6$$

$$P(X = x \mid E) = \frac{1}{3} \quad \text{FOR} \quad x = 2, 4, \text{ OR } 6$$

PMF

x	~~1~~	2	~~3~~	4	~~5~~	6
$p_X(x)$	~~1/6~~	1/6	~~1/6~~	1/6	~~1/6~~	1/6

x	2	4	6
$P(X=x \mid E)$	1/3	1/3	1/3

You flip 3 coins. What is the PMF number of heads given that there is $\underbrace{\text{at least one}}_{A}$?

UNCONDITIONAL PMF

x	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
$P(X=x \mid A)$	0	$\frac{3}{7}$	$\frac{3}{7}$	$\frac{1}{7}$

Binomial$(3, \frac{1}{2})$

$x = 1, 2,$ OR 3: $P(X=x \mid A) = \frac{P(X=x \text{ AND } A)}{P(A)} = \frac{P(X=x)}{P(X=1)+P(X=2)+P(X=3)}$

Conditioning on a random variable

Let X and Y be random variables.

The conditional PMF of X given Y is

JOINT

$$\mathbf{P}(X = x \mid Y = y) = \frac{\mathbf{P}(X = x \text{ and } Y = y)}{\mathbf{P}(Y = y)}$$

MARGINAL

$$p_{X|Y}(x \mid y) = \frac{p_{XY}(x, y)}{p_Y(y)}$$

For fixed y, $p_{X|Y}$ is a PMF as a function of x. $\sum_x p_{X|Y}(x|y) = 1$

FOR ALL y

Roll two 4-sided dice. What is the PMF of the sum given the first roll?

X = FIRST ROLL

S = SUM

$$P(S=s \mid X=x) = \frac{P(S=s, X=x)}{P(X=x)}$$

$P(X=x) = 1/4$

X \ S	2	3	4	5	6	7	8
1	1/4	1/4	1/4	1/4	0	0	0
2	0	1/4	1/4	1/4	1/4	0	0
3	0	0	1/4	1/4	1/4	1/4	0
4	0	0	0	1/4	1/4	1/4	1/4

$p_{S|X}(s|x)$

Roll two 4-sided dice. What is the PMF of the sum given the first roll?

JOINT

F \ S	2	3	4	5	6	7	8	MARGINAL $P(F=f)$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	0	0	0	$\frac{1}{4}$
2	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{1}{4}$
3	0	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	0	$\frac{1}{4}$
4	0	0	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$

COND. $P_{S|F}$

F \ S	2	3	4	5	6	7	8
1	1/4	1/4	1/4	1/4			
2		1/4	1/4	1/4	1/4		
3			1/4	1/4	1/4	1/4	
4				1/4	1/4	1/4	1/4

Roll two 4-sided dice. What is the PMF of the first roll given the sum?

F \ S	2	3	4	5	6	7	8	
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	0	0	0	$\frac{1}{4}$
2	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{1}{4}$
3	0	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	0	$\frac{1}{4}$
4	0	0	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$
	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	

MARGINAL

$P(F=f)$

$P(S=s)$

S \ F	1	2	3	4
2	1	0	0	0
3	1/2	1/2	0	0
4	1/3	1/3	1/3	0
5	1/4	1/4	1/4	1/4
6	0	1/3	1/3	1/3
7	0	0	1/2	1/2
8	0	0	0	1

Conditional Expectation

The conditional expectation of X given event A is

$$\mathbf{E}[X \mid A] = \sum_x x\,\mathbf{P}(X = x \mid A)$$

The conditional expectation of X given $Y = y$ is

$$\mathbf{E}[X \mid Y = y] = \sum_x x\,\mathbf{P}(X = x \mid Y = y)$$

You flip 3 coins. What is the expected number of heads given that there is at least one?

x	1	2	3
$P(X=x \mid A)$	$3/7$	$3/7$	$1/7$

$$E[X \mid A] = 1 \cdot \frac{3}{7} + 2 \cdot \frac{3}{7} + 3 \cdot \frac{1}{7} = \frac{12}{7}$$

Total Expectation Theorem

$$\boxed{\mathbf{E}[X] = \mathbf{E}[X \mid A]\,\mathbf{P}(A) + \mathbf{E}[X \mid A^c]\,\mathbf{P}(A^c)}$$

Proof

$$\sum_x x\,P(X=x|A)P(A) + \sum_x x\,P(X=x|A^c)P(A^c)$$

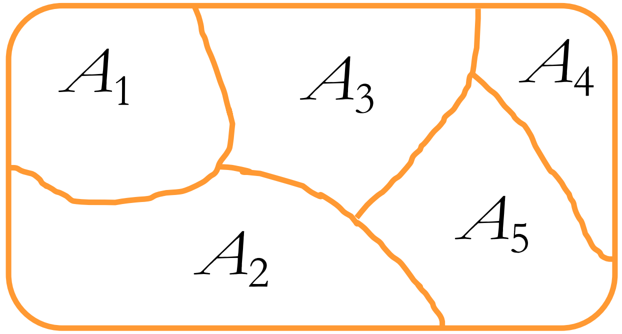
$$= \sum_x x\left(P(X=x|A)P(A) + P(X=x|A^c)P(A^c)\right)$$

$$= \sum_x x\,P(X=x)$$

$$= E[X]$$

Total Expectation Theorem (general form)

If $A_1, \ldots, A_n$ partition Ω
then



$$\mathbf{E}[X] = \mathbf{E}[X \mid A_1]\mathbf{P}(A_1) + \ldots + \mathbf{E}[X \mid A_n]\mathbf{P}(A_n)$$

In particular

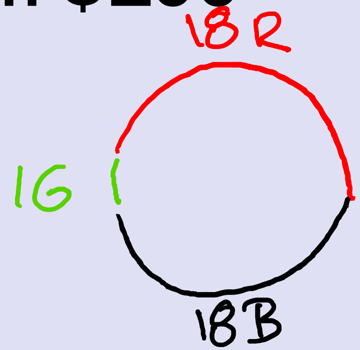
$$\mathbf{E}[X] = \sum_y \mathbf{E}[X \mid Y = y]\,\mathbf{P}(Y = y)$$

type	👩‍🏫	👵	🕵️‍♂️
average time on facebook	30 min	60 min	10 min
% of visitors	60%	30%	10%

average visitor time = $E[X|A]P(A) + E[X|B]P(B) + E[X|C]P(C) = 30 \cdot 60\% + 60 \cdot 30\% + 10 \cdot 10\%$

You play 10 rounds of roulette. You start with \$100 and bet 10% on red in every round.

On average, how much cash will remain?



X_t = CASH AFTER t ROUNDS

$$E[X_1] = E[X_1 | W_1] P(W_1) + E[X_1 | W_1^c] P(W_1^c)$$
$$= 110 \cdot \frac{18}{37} + 90 \cdot \frac{19}{37}$$

$$E[X_t] = E[X_t | W_t] P(W_t) + E[X_t | W_t^c] P(W_t^c)$$
$$= E[1.1 X_{t-1}] \frac{18}{37} + E[0.9 X_{t-1}] \cdot \frac{19}{37}$$
$$= \left(1.1 \cdot \frac{18}{37} + 0.9 \cdot \frac{19}{37}\right) \cdot E[X_{t-1}]$$
$$\approx 0.997 E[X_{t-1}]$$

$$E[X_{10}] \approx 0.997^{10} \cdot 100$$
$$\approx 97.3$$

You flip 3 coins. What is the expected number of heads given that there is at least one?

X = NUMBER OF HEADS

A = AT LEAST ONE

$$E[X] = E[X|A]P(A) + E[X|A^c]P(A^c)$$

$$\frac{3}{2} = E[X|A] \cdot \frac{7}{8} + 0 \cdot \frac{1}{8}$$

$$E[X|A] = \frac{3/2}{7/8} = \frac{12}{7}.$$

Mean of the Geometric

$X = \text{Geometric}(p)$ **random variable**

$$\overbrace{F F \ldots F S}^{X}, \quad F\underbrace{F \ldots F S}_{Y}$$

$X = 1 + Y$

Y ALSO $\text{Geom}(p)$ <u>GIVEN A^c</u>

$\mathbf{E}[X] = p \cdot 1 + (1-p) \cdot p \cdot 2 + \ldots$

<u>ANOTHER WAY:</u>

A = TRIAL 1 SUCCEEDS

$$E[X] = E[X|A]P(A) + E[X|A^c]P(A^c)$$
$$= 1 \cdot p + E[1+Y|A^c] \cdot (1-p)$$
$$= p + (1+E[Y|A^c]) \cdot (1-p)$$
$$= p + (1+E[X]) \cdot (1-p)$$

$$\boxed{E[X] = 1/p}$$

Variance of the Geometric

$X = \text{Geometric}(p)$ **random variable**

$$\mathbf{Var}[X] = E\left[\left(X-\tfrac{1}{p}\right)^2\right]$$

$$= E\left[\left(X-\tfrac{1}{p}\right)^2 \middle| A\right]\cdot p + E\left[\left(X-\tfrac{1}{p}\right)^2 \middle| A^c\right]\cdot(1-p)$$

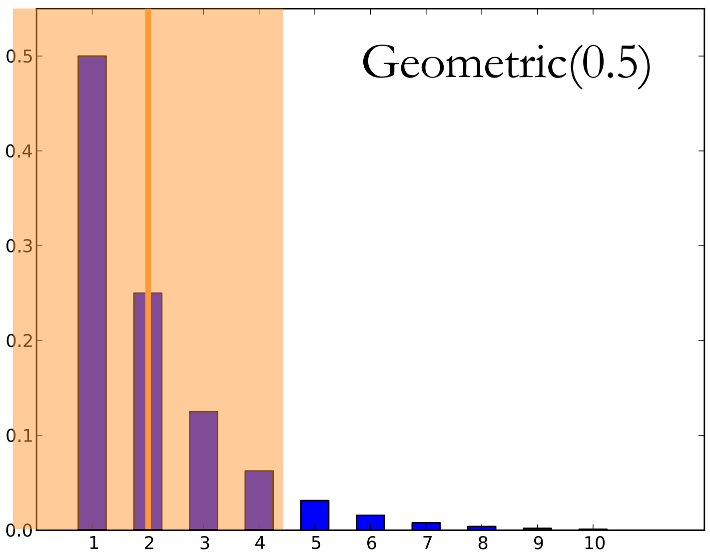
$$= \left(1-\tfrac{1}{p}\right)^2\cdot p + E\left[\left(1+Y-\tfrac{1}{p}\right)^2 \middle| A^c\right]\cdot(1-p)$$

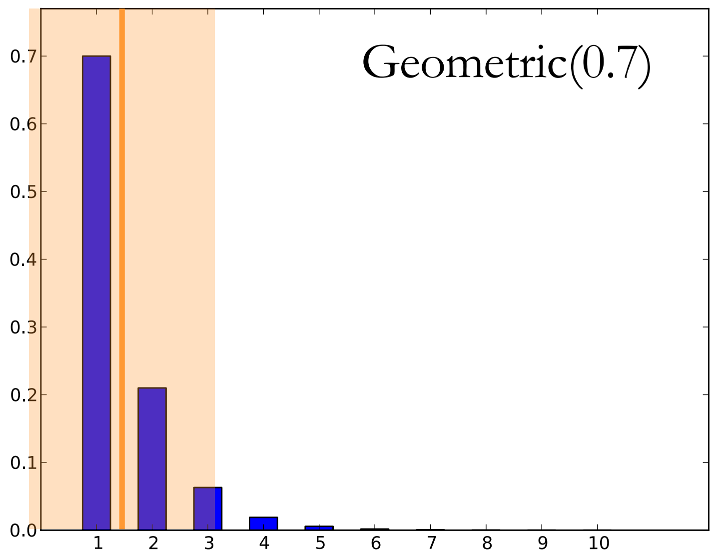
$$= \left(1-\tfrac{1}{p}\right)^2\cdot p + E\left[1+2\left(Y-\tfrac{1}{p}\right)+\left(Y-\tfrac{1}{p}\right)^2 \middle| A^c\right](1-p)$$

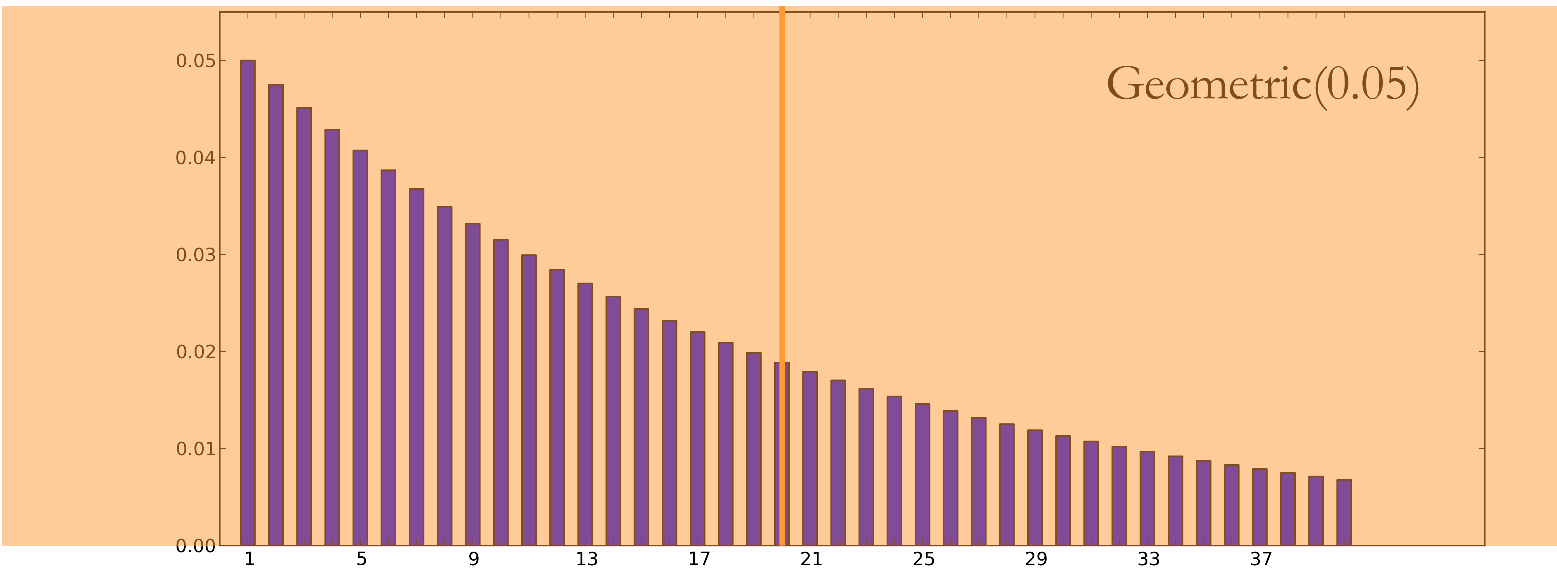
$$= \left(1-\tfrac{1}{p}\right)^2\cdot p + \left(1+2\underbrace{\cancel{E\left[Y-\tfrac{1}{p} \middle| A^c\right]}}_{0}+\underbrace{E\left[\left(Y-\tfrac{1}{p}\right)^2 \middle| A^c\right]}_{\mathrm{Var}[X]}\right)(1-p)$$

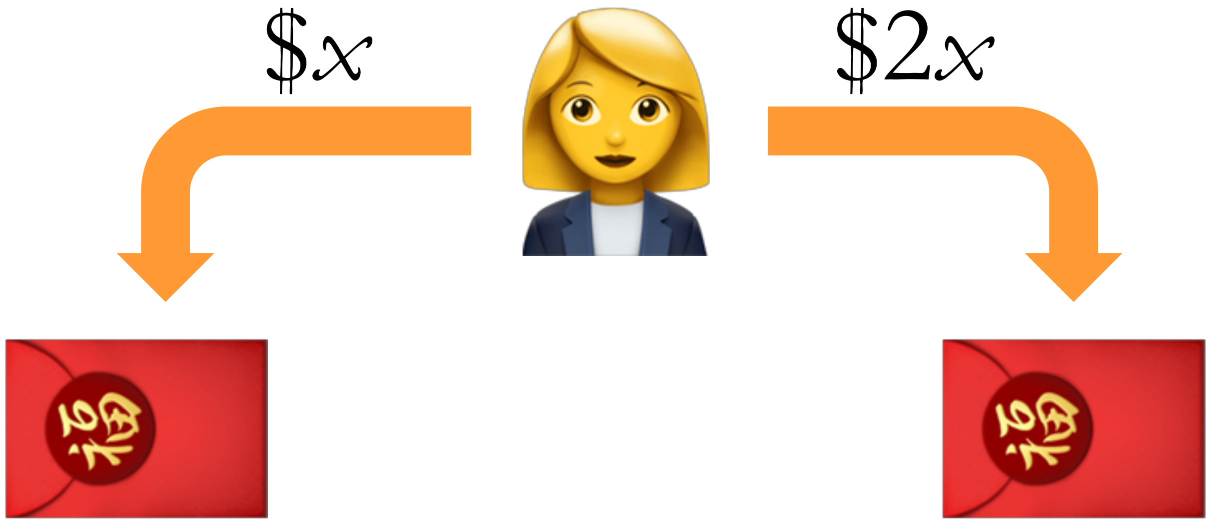
$$= \left(1-\tfrac{1}{p}\right)^2\cdot p + \left(1+\mathrm{Var}[X]\right)\cdot(1-p)$$

$$\boxed{\mathrm{Var}[X] = \left(\tfrac{1}{p}-1\right)^2+\left(\tfrac{1}{p}-1\right) = \frac{1-p}{p^2}}$$


Geometric(0.5)
0.0
0.1
0.2
0.3
0.4
0.5
1 2 3 4 5 6 7 8 9 10


Geometric(0.7)
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
1 2 3 4 5 6 7 8 9 10


Geometric(0.05)
0.00
0.01
0.02
0.03
0.04
0.05
1 5 9 13 17 21 25 29 33 37



stay or switch?

Bob should stay because...

DON'T LOSE WHAT YOU HAVE

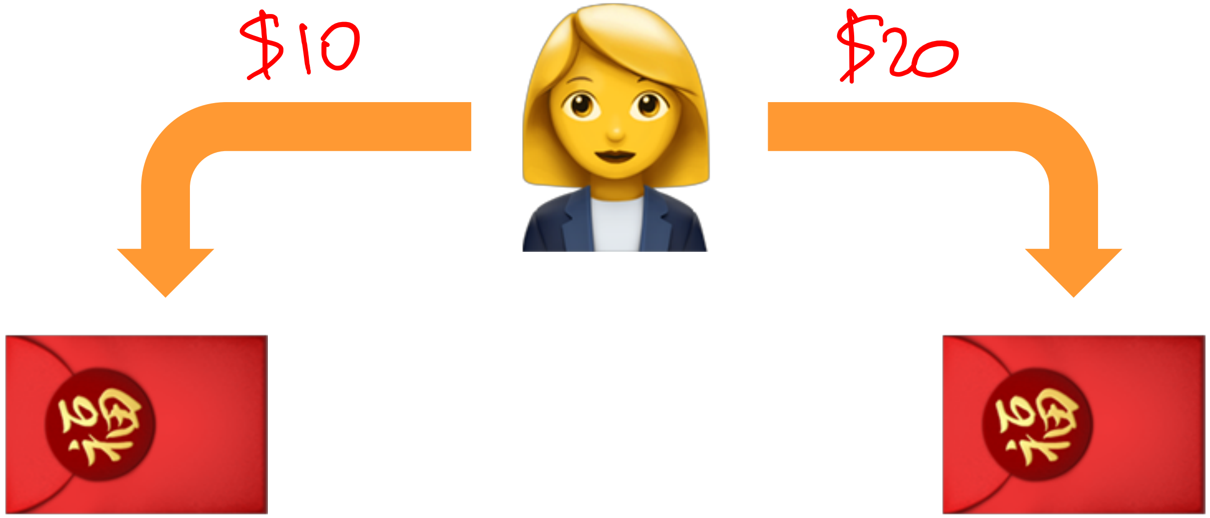
DOESN'T MAKE A DIFFERENCE

Bob should switch because...

X STAY

Y SWITCH

$$
\begin{aligned}
E[Y] &= \frac{1}{2}E[2X] + \frac{1}{2}E[X/2] \\
&= \frac{1}{2}\cdot 2E[X] + \frac{1}{2}\cdot\frac{1}{2}E[X] \\
&= \frac{5}{4}E[X]
\end{aligned}
$$

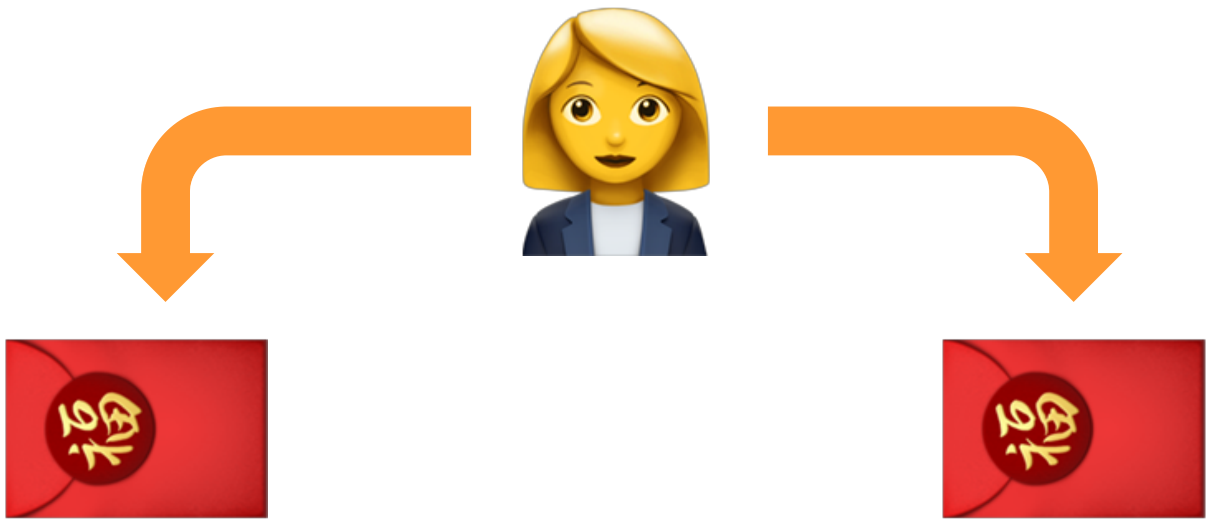


	X	Y
1/2	\$10	\$20
1/2	\$20	\$10

$E[X] = 15$ $E[Y] = 15$

SWITCH IF \$10
STAY IF \$20

$\boxed{\$20}$ ALWAYS!



$\frac{1}{2}$ \$10

$\frac{1}{2}$ \$20

\$20

\$40

$$E[Y \mid X = 20]$$
$$= \frac{1}{2} \cdot 10 + \frac{1}{2} \cdot 40$$
$$= 25$$

SWITCH!

JOINT PMF

X	Y	P(X=x, Y=y)
10	20	1/4
20	10	1/4
20	40	1/4
40	20	1/4

Independent random variables

Let X and Y be discrete random variables.

X and Y are independent if

$$\mathbf{P}(X = x, Y = y) = \mathbf{P}(X = x)\,\mathbf{P}(Y = y)$$

for all possible values of x and y.

In PMF notation, $\mathrm{p}_{XY}(x, y) = p_X(x)\,p_Y(y)$ for all x, y.

Independent random variables

X and Y are **independent** if

$$\mathbf{P}(X = x \mid Y = y) = \mathbf{P}(X = x)$$

for all x and y such that $\mathbf{P}(Y = y) > 0$.

In PMF notation, $p_{X|Y}(x \mid y) = p_X(x)$ if $p_Y(y) > 0$.

Alice tosses 3 coins and so does Bob. Alice gets \$1 per head and Bob gets \$1 per tail.

Are their earnings independent?

YES

$$P(B=b \mid A=a) = P(B=b) = \binom{3}{b} \cdot \frac{1}{8}.$$

Now they toss the same coin 3 times. Are their earnings independent?

$$A + B = 3$$

$$\left.\begin{aligned} &P(B=3 \mid A=3) = 0 \\ &P(B=3) = \frac{1}{8} \end{aligned}\right\} \quad \underline{\text{DEPENDENT}}$$

Expectation and independence

X and Y are independent if and only if

$$\mathbf{E}[f(X)g(Y)] = \mathbf{E}[f(X)]\ \mathbf{E}[g(Y)]$$

for all real valued functions f and g.

Expectation and independence

In particular, if X and Y are independent then

$$E[XY] = E[X]\,E[Y]$$

Not true in general!

Variance of a sum

Recall $\mathbf{Var}[X] = \mathbf{E}[(X - \mathbf{E}[X])^2] = \mathbf{E}[X^2] - \mathbf{E}[X]^2$

$$\mathbf{Var}[X + Y] = E[(X+Y)^2] - E[X+Y]^2$$

$$E[X^2 + 2XY + Y^2]$$
$$= E[X^2] + E[2XY] + E[Y^2]$$
$$= E[X^2] + 2E[X]E[Y] + E[Y^2]$$

$$(E[X] + E[Y])^2$$
$$= E[X]^2 + E[Y]^2 + 2E[X]E[Y]$$

$$= \mathrm{Var}[X] + \mathrm{Var}[Y]$$

Variance of a sum

$$\mathbf{Var}[X_1 + \ldots + X_n] = \mathbf{Var}[X_1] + \ldots + \mathbf{Var}[X_n]$$

if every pair X_i, X_j is independent.

Not true in general!

Variance of the Binomial

Binomial(n, p)

$$X = X_1 + X_2 + \dots + X_n$$

1 IF TRIAL SUCCEEDS
0 IF TRIAL FAILS

$$Var[X] = Var[X_1] + Var[X_2] + \dots + Var[X_n]$$

$$Var[X_1] = E[X_1^2] - E[X_1]^2 =$$

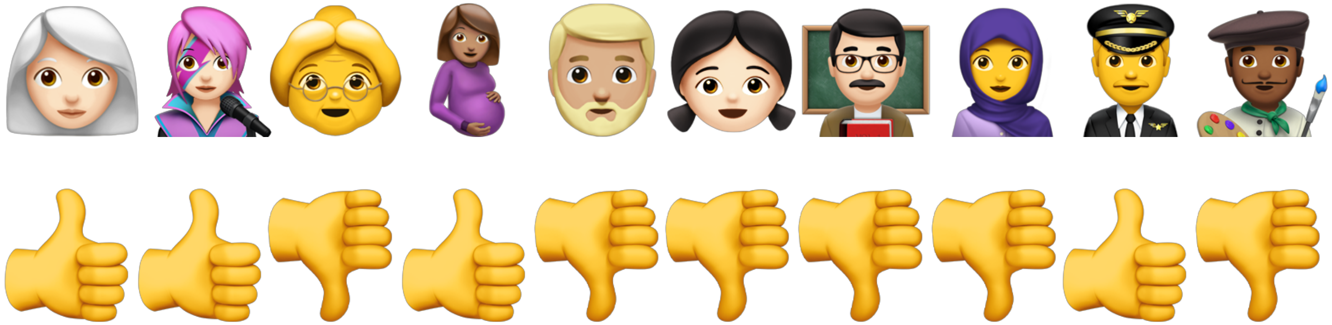
$$= p - p^2$$

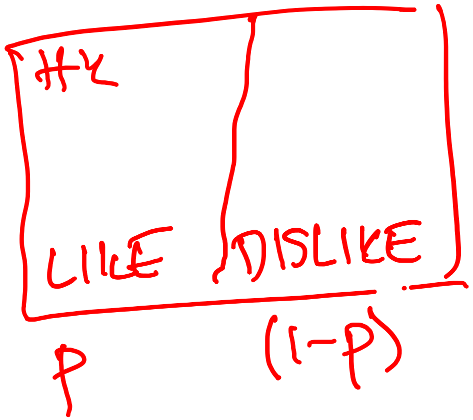
$$= p(1-p)$$

$$Var[X] = n \cdot p \cdot (1-p)$$

Sample mean

40%





X = # POLLED PEOPLE THAT LIKE

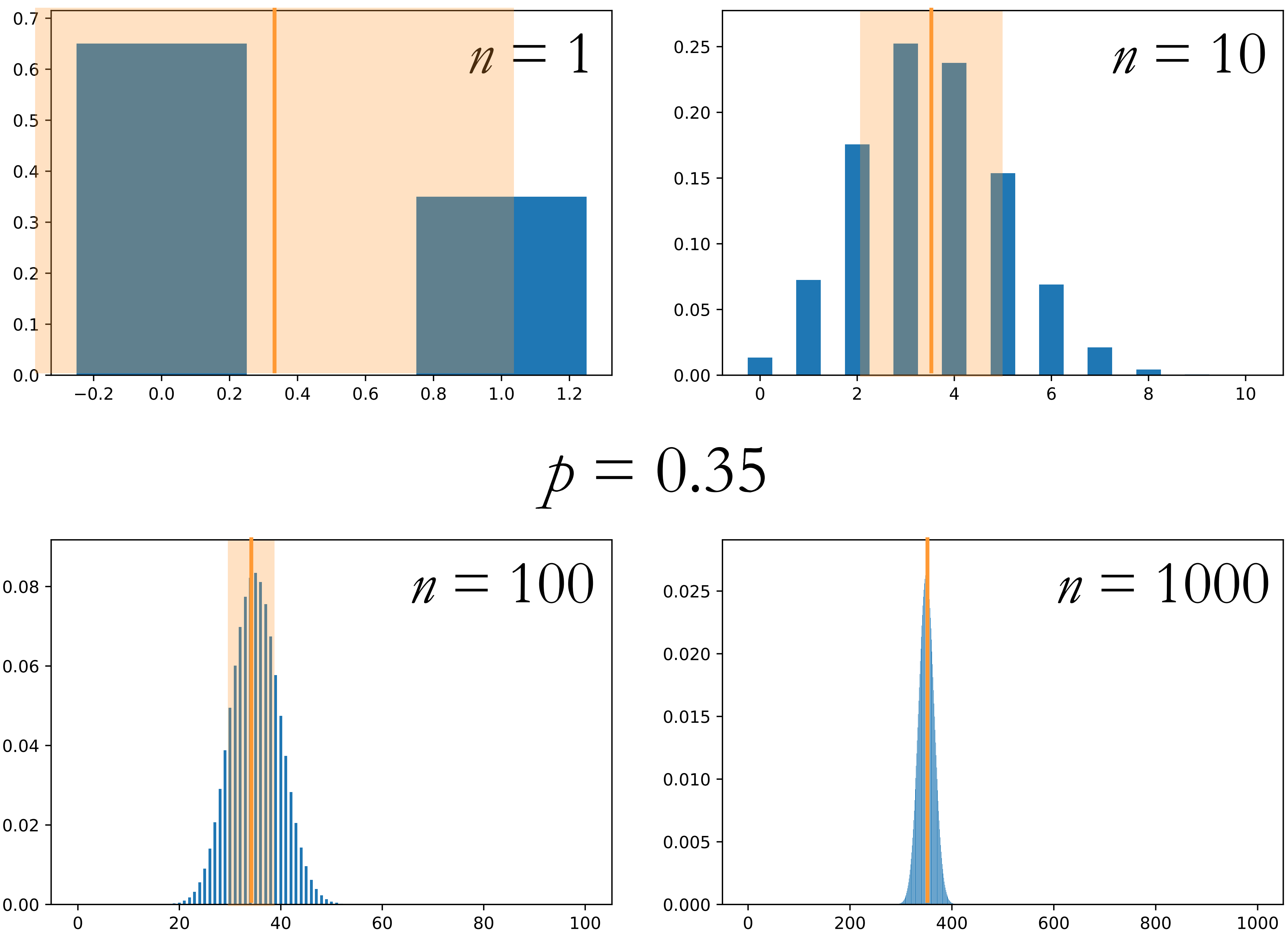
$\text{Binomial}(n, p)$

$$E[X] = n \cdot p$$

$$\sigma = \sqrt{n \cdot p(1-p)}$$

Ex. $p = 50\%$ $\quad E[X] = \frac{n}{2} \quad \sigma = \frac{\sqrt{n}}{2}$

	$E[X]$	σ
$n = 100$	50	5
$n = 10000$	5000	50



$n = 1$
$n = 10$
$p = 0.35$
$n = 100$
$n = 1000$

Variance of the Poisson

Poisson(λ) **approximates** Binomial(n, λ/n) **for large** n

$$\boxed{p(k) = e^{-\lambda} \lambda^k / k!} \qquad k = 0, 1, 2, 3, \ldots$$

$$\mathrm{Var}[\mathrm{Binomial}(n,p)] = n \cdot p \cdot (1-p)$$

$$\lim_{n\to\infty} n \cdot p \cdot (1-p) = \lim_{n\to\infty} n \cdot \frac{\lambda}{n}\left(1-\frac{\lambda}{n}\right) = \lambda \qquad \left(p = \frac{\lambda}{n}\right)$$

$$\mathrm{Var}[\mathrm{Poisson}(\lambda)] = \lambda \qquad \sigma = \sqrt{\lambda}$$

Independence of multiple random variables

X, Y, Z independent if

$$\mathbf{P}(X = x, Y = y, Z = z) = \mathbf{P}(X = x)\, \mathbf{P}(Y = y)\, \mathbf{P}(Z = z)$$

for all possible values of x, y, z.

X, Y, Z independent if and only if

$$\mathbf{E}[f(X)g(Y)h(Z)] = \mathbf{E}[f(X)]\, \mathbf{E}[g(Y)]\, \mathbf{E}[h(Z)]$$

for all f, g, h.

Usual warnings apply.