Bayesian Statistics

2 decks of cards

1 deck Red deck: 28 red cards
24 black cards

Black deck: 28 black cards
24 red cards

R B B R R B R

conditioned on "red deck" what is the prob of this sequence $\left(\frac{28}{52}\right)^4\left(\frac{24}{52}\right)^3$

conditioned on "black deck" what is the prob. of this sequence $\left(\frac{24}{52}\right)^4\left(\frac{28}{52}\right)^3$

→

Let $\theta = 0$ if "red deck"
$= 1$ if "black deck"

$X_1, \ldots, X_n$ $\qquad P(x_1, \ldots, x_7 \mid \theta = 0) = \left(\frac{28}{52}\right)^4\left(\frac{24}{52}\right)^3$

$$p(x_1, \ldots, x_7 \mid \theta = 1) = \left(\frac{24}{52}\right)^4\left(\frac{28}{52}\right)^3$$

$$P(\theta = 0 \mid x_1, \ldots, x_7) = \frac{P(\theta=0, x_1, \ldots, x_7)}{P(x_1, \ldots, x_7)}$$

$$= \frac{P(\theta=0, x_1, \ldots, x_7)}{P(\theta=0, x_1, \ldots, x_7) + P(\theta=1, x_1, \ldots, x_7)}$$

$$= \frac{P(\theta=0)\, P(x_1, \ldots, x_7 \mid \theta=0)}{P(\theta=0)\, P(x_1, \ldots, x_7 \mid \theta=0) + P(\theta=1) P(x_1, \ldots, x_7 \mid \theta=1)}$$

$P(\theta = 0) = \frac{1}{2}$

$$P(\theta = 0 \mid x_1, \ldots, x_7) = \frac{(28)^4 (24)^3 \cdot \frac{1}{2}}{\frac{1}{2}(28)^4(24)^3 + (28)^3(24)^4 \frac{1}{2}}$$

$$= \frac{28}{52}$$

$$P(\theta = 1 \mid x_1, \ldots, x_7) = \frac{24}{52}$$

R B B R R B R B B

$$P(\theta = 0 \mid x_1, \ldots, x_9) = \frac{(28)^4 (24)^5}{(28)^4(24)^5 + (28)^5(24)^4}$$

$$= \frac{24}{52}$$

$$f_{\theta|X}(\theta|x) = \frac{f(x,\theta)}{f(x)} = \frac{f(\theta)\, f(x|\theta)}{f(x)}$$

$$\frac{f_{\theta|X}(\theta = i|x)}{f_{\theta|x}(\theta = j|x)} = \frac{f(\theta = i)\; f(x|\theta = i)}{f(\theta = j)\; f(x|\theta = j)}$$

if conditioned on θ, $X_1, \ldots, X_n$ are independent

$$f(x_1, \ldots, x_n \mid \theta) = \prod_{k=1}^{n} f(x_k|\theta)$$

$$\therefore \frac{f_{\theta|X}(\theta = i|\vec{x})}{f_{\theta|X}(\theta = j|\vec{x})} = \frac{f(\theta = i)\quad f(x_1, \ldots, x_n|\theta = i)}{f(\theta = j)\quad f(x_1, \ldots, x_n|\theta = j)}$$

$$= \frac{f(\theta=i)\prod_k f(x_k|\theta=i)}{f(\theta=j)\prod_k f(x_k|\theta=j)}$$

$$= \frac{f(\theta=i)}{f(\theta=j)}\prod_k \frac{f(x_k|\theta=i)}{f(x_k|\theta=j)}$$

$\theta=1$ uniform $[0, 18 \text{ minutes})$

$\theta=2$ uniform $[0, 48 \text{ minutes})$

$\theta=3$ uniform $(0, 36 \text{ minutes})$

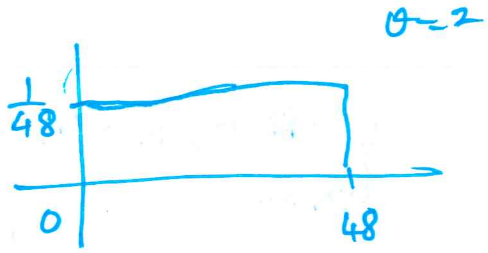
$x_1 = 30$ minutes

Prior $\rightarrow$ $P(\theta=1) = P(\theta=2) = P(\theta=3) = \frac{1}{3}$

$$P(\theta=1 | x_1 = 30\text{minutes}) = \frac{P[\theta=1, x_1=30]}{P[x_1=30]} = 0$$

~~P[θ=2~~, $f(x_1=30|\theta=2) = \frac{1}{48}$

$f(x_1=30|\theta=3) = \frac{1}{36}$



$$\frac{P(\theta=2|x_1=30)}{P(\theta=3|x_1=30)} = \frac{\frac{1}{48}}{\frac{1}{36}} = \frac{36}{48}$$

$$P(\theta=2|x_1=30) = \frac{P(\theta=2)\,P(x_1=30|\theta=2)}{P(\theta=3)\,P(x_1=30|\theta=3) + P(\theta=2)\times P(x=30|\theta=2)}$$

$$= \frac{\frac{1}{48}}{\frac{1}{48}+\frac{1}{36}} = \frac{36}{84} = \frac{3}{7}$$

$P(\theta=3|x_1=30) = \frac{4}{7}$

Puzzle Two envelopes, one contains number x, Another contains number y. don't know x or y.

Goal. Open an envelope look ~~atte~~ at the value and then decide whether to switch or not?

Qn: Can you devise a strategy that beats an expected value of $\frac{x+y}{2}$.

Inference for Normals

$\cancel{f(\Theta|X_1=x_1,\ldots,X_n=x_n)}$ $\quad f_{\Theta|X_1,X_2}(\theta|X_1=x_1, X_2=x_2)$

$$= \frac{\cancel{f(\theta, X_1=x_1,}\; f_{\Theta X_1 X_2}(\Theta=\theta, X_1=x_1, X_2=x_2)}{f_{X_1,X_2}(X_1=x_1, X_2=x_2)}$$

$$f_{\Theta,X_1,X_2}(\Theta=\theta, X_1=x_1, X_2=x_2) = f_{\Theta}(\Theta=\theta)\, f_{X_1X_2|\Theta}(X_1=x_1, X_2=x_2|\Theta=\theta)$$

$$= \frac{1}{\sqrt{2\pi\sigma_0^2}}\, e^{-\frac{1}{2\sigma_0^2}(\theta-x_0)^2}\; f_{X_1|\Theta}(X_1=x_1|\Theta=\theta)\, f_{X_2|\Theta}(X_2=x_2|\Theta=\theta)$$

$$= \frac{1}{\sqrt{2\pi\sigma_0^2}}\, e^{-\frac{(\theta-x_0)^2}{2\sigma_0^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_1^2}}\, e^{-\frac{(\theta-x_1)^2}{2\sigma_1^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_2^2}}\, e^{-\frac{(\theta-x_2)^2}{2\sigma_2^2}}$$

$$f(X_1 = x_1,\ X_2 = x_2) = \int_{-\infty}^{\infty} f(\Theta = \theta,\ X_1 = x_1,\ X_2 = x_2)\, d\theta$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\left[\frac{\theta^2}{\sigma_0^2} + \frac{\theta^2}{\sigma_1^2} + \frac{\theta^2}{\sigma_2^2} - \frac{2\theta x_0}{\sigma_0^2} - \frac{2\theta x_1}{\sigma_1^2} - \frac{2\theta x_2}{\sigma_2^2}\right]}}{(2\pi)^{3/2}\sqrt{\sigma_0^2\sigma_1^2\sigma_2^2}} \times e^{-\frac{1}{2}\left[\frac{x_0^2}{\sigma_0^2} + \frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2}\right]}\, d\theta$$

$$= \frac{e^{-\frac{1}{2}\left[\frac{x_0^2}{\sigma_0^2} + \frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2}\right]}}{(2\pi)^{3/2}\sqrt{\sigma_0^2\sigma_1^2\sigma_2^2}} \left[\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\theta^2\left(\frac{1}{\sigma_0^2} + \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right) - 2\left(\frac{\theta x_0}{\sigma_0^2} + \frac{\theta x_1}{\sigma_1^2} + \frac{\theta x_2}{\sigma_2^2}\right)\right]}\, d\theta\right]$$

$$= \frac{e^{-\frac{1}{2}\left[\frac{x_0^2}{\sigma_0^2} + \frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2}\right]}}{(2\pi)^{3/2}\sqrt{\sigma_0^2\sigma_1^2\sigma_2^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{\left[\theta - \frac{\left(\frac{x_0}{\sigma_0^2} + \frac{x_1}{\sigma_1^2} + \frac{x_2}{\sigma_2^2}\right)}{\left(\frac{1}{\sigma_0^2} + \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)}\right]^2}{\left(\frac{1}{\frac{1}{\sigma_0^2} + \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}\right)}}$$

$$\times\ e^{-\frac{1}{2}\frac{\left(\frac{x_0}{\sigma_0^2} + \frac{x_1}{\sigma_1^2} + \frac{x_2}{\sigma_2^2}\right)^2}{\left(\frac{1}{\sigma_0^2} + \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)}}\, d\theta$$

$$= \sqrt{2\pi\frac{1}{\left(\frac{1}{\sigma_0^2} + \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)}} \times \frac{e^{-\frac{1}{2}\left(\frac{x_0^2}{\sigma_0^2} + \frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2}\right)}}{(2\pi)^{3/2}\sqrt{\sigma_0^2\sigma_1^2\sigma_2^2}} \times$$

$$e^{-\frac{1}{2}\frac{\left(\frac{x_0}{\sigma_0^2} + \frac{x_1}{\sigma_1^2} + \frac{x_2}{\sigma_2^2}\right)^2}{\frac{1}{\sigma_0^2} + \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}}$$

$$f(\Theta=\theta \mid X_1=x_1, X_2=x_2) = \frac{f(\Theta=\theta, X_1=x_1, X_2=x_2)}{f(X_1=x_1, X_2=x_2)}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(\theta - x)^2}{\sigma^2}}$$

where $\frac{1}{\sigma^2} = \frac{1}{\sigma_0^2} + \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$

and $\frac{x}{\sigma^2} = \frac{x_0}{\sigma_0^2} + \frac{x_1}{\sigma_1^2} + \frac{x_2}{\sigma_2^2}$

Two coins

HHTH

C1: $Pr(H) = \frac{1}{3}$

C2: $Pr(T) = \frac{1}{3}$

what is the prior that yields

$$P(\theta=1 \mid HHTH) = P(\theta=2 \mid HHTH)$$

Let $P(\theta=1) = x$

then $P(\theta=1, HHTH) = x \cdot \left(\frac{1}{3}\right)^3 \frac{2}{3}$

$P(\theta=2, HHTH) = (1-x)\left(\frac{2}{3}\right)^3 \frac{1}{3}$

$$P(HHTH) = x\left(\frac{1}{3}\right)^3 \cdot \frac{2}{3} + (1-x)\left(\frac{2}{3}\right)^3 \frac{1}{3}$$

want $P(\theta=1 \mid HHTH) = P(\theta=2 \mid HHTH)$

$\iff P(\theta=1, HHTH) = P(\theta=2, HHTH)$

$\therefore x\left(\frac{1}{3}\right)^3 \frac{2}{3} = (1-x)\left(\frac{2}{3}\right)^3 \frac{1}{3}$

$\Rightarrow 2x = (1-x)\, 8 \iff \frac{x}{1-x} = 4$

or $x = \frac{4}{5}$

Examples

Let us consider 3 die.

$P(D=1) = \frac{1}{4}$, $P(D=2) = \frac{1}{3}$, $P(D=3) = \frac{5}{12}$

~~Prob~~ (when $D=1$, prob of $\{1, \ldots, 6\}$ are $\{\frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}, 0\}$

when $D=2$, prob of $\{1, \ldots, 6\}$ are $\{\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{4}\}$

when $D=3$, prob of $\{1, 2, \ldots, 6\}$ are $\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}$

updated prior after seeing (8)

3 2 6 2 4

$$P(\theta = 1 \mid 3,2,6,2,4) = 0$$

$$P(\theta = 2 \mid 3,2,6,2,4) \propto \frac{1}{3} \times \left(\frac{1}{8}\right)^4 \times \frac{1}{4} = \frac{\frac{1}{3\times 8^4 \times 4}}{\frac{1}{3\times 8^4\times 4} + \frac{5}{6^6 \times 2}}$$

$$P(\theta = 3 \mid 3,2,6,2,4) \propto \frac{5}{12} \times \left(\frac{1}{6}\right)^5$$

$$= \frac{\frac{1}{3072}}{\frac{1}{3072} + \frac{5}{27\times 16}} = \frac{243}{883}$$

	new
$\frac{1}{4}$	0
$\frac{1}{3}$	$\frac{243}{883}$
$\frac{5}{12}$	$\frac{640}{883}$

Maximum Likelihood Estimator $\quad \arg\max_{\theta} f_{X|\theta}(x|\theta)$

VS

MAP Estimator $\quad \arg\max_{\theta} f_{\theta|X}(\theta|x)$