Practice questions

1. Roll a die twice. Let X be the larger number and Y be the smaller number you rolled. Find the joint PMF of X and Y, their marginal PMFs, and the expected value of $X + Y$.

   Solution: When $x > y$ are different, the event $X = x$ and $Y = y$ can happen in 2 out of 36 possible ways: Either the first toss is x and the second toss is y, or the other way around. When $x = y$ then there is only one possible outcome. All other probabilities are zero. Summarizing, the joint PMF $p_{XY}(x, y)$ is

$x\backslash y$	1	2	3	4	5	6
1	1/36	0	0	0	0	0
2	1/18	1/36	0	0	0	0
3	1/18	1/18	1/36	0	0	0
4	1/18	1/18	1/18	1/36	0	0
5	1/18	1/18	1/18	1/18	1/36	0
6	1/18	1/18	1/18	1/18	1/18	1/36

   The marginal PMFs are obtained by adding the colums and rows, respectively:

z	1	2	3	4	5	6
$p_X(z)$	1/36	3/36	5/36	7/36	9/36	11/36
$p_Y(z)$	11/36	9/36	7/36	5/36	3/36	1/36

   The expected values are $\mathrm{E}[X] = 161/36$ and $\mathrm{E}[Y] = 91/36$. The expected sum is $\mathrm{E}[X+Y] = \mathrm{E}[X] + \mathrm{E}[Y] = 7$. Another way to see this is that if A and B are the first and second rolls then $X + Y = A + B$, so $\mathrm{E}[X + Y] = \mathrm{E}[A + B] = \mathrm{E}[A] + \mathrm{E}[B] = 2 \cdot 7/2 = 7$.

2. A number between 1 to 100 is selected at random. You want to guess the number by asking questions of the type "Is the number equal to x"? For every incorrect guess you lose \$1, and for the correct one you get \$40. You play this game until you make a correct guess. What is your expected profit?

   Solution: Let random variable X denote the number of guesses you make (including the final correct guess). The probability that $X = 1$ is just the chance that you guess correctly the first time, or $1/100$. The probability that $X = 2$ is $99/100 \cdot 1/99 = 1/100$. Similarly, for any $i \in \{1, 2, ..., 100\}$,

$$\Pr[X = i] = \frac{99}{100} \cdot \frac{98}{99} \cdots \frac{100 - i + 1}{100 - i + 1} \cdot \frac{1}{100 - i + 1} = \frac{1}{100}.$$

   Thus, the expected value of X is $\frac{1}{100} \cdot (1 + 2 + ... + 100) = 50.5$.

   The payout P equals $40 - (X - 1)$, so the expected payout is

$$\mathrm{E}[P] = \mathrm{E}[40 - (X - 1)] = 41 - \mathrm{E}[X] = 41 - 50.5 = -9.5.$$

3. On any given day between Monday and Saturday, the probability that you'll have a late snack is 20%, independent on the other days. You'll have a late snack on Sunday *if and only if* you

didn't have one in the previous six days. What is the expected number of snacks you'll be having?

Solution: Let X be the number of snacks you have between Monday and Saturday. Then X is a Binomial$(6, 0.2)$ random variable. The number of snacks you will be having in the whole week is

$$Y = \begin{cases} X, & \text{if } X > 0 \\ 1, & \text{if } X = 0. \end{cases}$$

As Y is a function of X, we can use the formula for the expectation of the function of a random variable to obtain

$$\mathrm{E}[Y] = 1\,\mathrm{P}(X = 1) + 2\,\mathrm{P}(X = 2) + \cdots + 6\,\mathrm{P}(X = 6) + 1\,\mathrm{P}(X = 0).$$

Using the binomial PMF formula $\mathrm{P}(X = x) = \binom{6}{x} \cdot 0.8^{6-x} \cdot 0.2^x$, we get that $\mathrm{E}[Y] \approx 1.46$.

4. Let p be a number between 0 and 1. Toss a p-biased coin. If the coin comes up heads, toss a fair coin and report the outcome twice (1 for heads, 0 for tails). If the coin comes up tails, report the outcomes of two independent fair coin tosses. Show that the marginal PMFs of your two reports are the same for every p, but the joint PMFs are all different.

   Solution: Let X and Y be the two reports and E be the event that the biased coin comes out heads (which occurs with probability p). We calculate the joint PMF of X and Y using the total probability formula:

   $$\mathrm{P}(X = x, Y = y) = \mathrm{P}(X = x, Y = y | E) \cdot p + \mathrm{P}(X = x, Y = y | E^c) \cdot (1 - p).$$

   If E occurs, the event "$X = x$ and $Y = y$" can never occur if $x \neq y$. Otherwise, $X = 1, Y = 1$ and $X = 0, Y = 0$ both occur with probability half. If E does not occur then the events $X = x$ and $Y = y$ are conditionally independent so $\mathrm{P}(X = x, Y = y | E^c) = 1/4$, so

   $$\mathrm{P}(X = x, Y = y) = \begin{cases} \frac{1}{2}p + \frac{1}{4}(1 - p), & \text{if } x = y \\ \frac{1}{4}(1 - p), & \text{if } x \neq y. \end{cases}$$

   The joint PMF of X and Y is:

$x \backslash y$	0	1
0	$\frac{1+p}{4}$	$\frac{1-p}{4}$
1	$\frac{1-p}{4}$	$\frac{1+p}{4}$

   and so the joint PMFs are all different as the value of p changes. On the other hand, the marginal PMFs, which equal the row/column sums, are uniform ($\mathrm{P}(X = 0) = \mathrm{P}(X = 1) = \mathrm{P}(Y = 0) = \mathrm{P}(Y = 1) = 1/2$) and therefore the same for all p.

5. On a given day, your golf score takes values from the range 101 to 110 with probability 0.1, independent of other days. To improve your score, you decide to play three times in a row and declare your score X to be the minimum of those scores. Calculate the PMF of X. By how much has your expected score improved as a result of playing on three days? *(Textbook problem 2.5.26)*

   Solution: We first calculate the PMF of X. For your lowest score to be 110, you must have scored this value on all three days, so $\mathrm{P}(X = 110) = 0.1^3$. For your lowest score to be *at least* 109, you must have scored 109 or 110 on all three days, so $\mathrm{P}(X \geq 109) = 0.2^3$. For the same reasons,

   $$\mathrm{P}(X \geq x) = (0.1 \cdot (111 - x))^3$$

for every x between 101 and 110. By the axioms, the probability that you get a score of *exactly* x is

$$\mathrm{P}(X = x) = \mathrm{P}(X \geq x) - \mathrm{P}(X \geq x - 1) = \frac{(111 - k)^3}{10^3} - \frac{(110 - k)^3}{10^3}$$

for every k between 101 and 110.

Originally, the score is uniformly distributed over 101,102,...,110 and thus the expected score is $\frac{1}{10}(101 + 102 + ... + 110) = 105.5$. Using the PMF of X given above, one can easily compute that the expected score now is 103.025; thus the expected score has improved by 2.475.