Practice Midterm 1

1. A coin is tossed 4 times. What is the probability there are at least 3 consecutive heads?

   Solution: The sample space is the set $\{\mathtt{H}, \mathtt{T}\}^4$. The event E of interest consists of the four sequences $\mathtt{HHHT}, \mathtt{THHH}, \mathtt{HHHH}$. All outcomes are equally likely, so $\mathrm{P}(E) = 3/2^4 = 3/16$.

2. When a coin comes out heads, you win \$1. When it comes out tails, you lose \$2. You toss the coin twice. Find the probability mass function and the expected value of your profit.

   Solution: Let X be the amount of money you gain. Then the probability mass function of X is

x:	-4	-1	2
$f(x)$:	$1/4$	$1/2$	$1/4$

   Therefore $\mathrm{E}[X] = 2 \cdot 1/4 + (-4) \cdot 1/4 + (-1) \cdot 1/2 = -1$.

3. Half the students know the answer to a true-false question. The other half guesses at random. I ask a random student and his answer is correct. What is the probability he knows the answer?

   Solution: The sample space consists of all students under equally likely outcomes. Let K be the event a student knows the answer, C be the event his answer is correct. We have $\mathrm{P}(K) = 1/2, \mathrm{P}(C \mid K) = 1, \mathrm{P}(C \mid K^c) = 1/2$. By Bayes' rule

   $$\mathrm{P}(K \mid C) = \frac{\mathrm{P}(C \mid K)\,\mathrm{P}(K)}{\mathrm{P}(C \mid K)\,\mathrm{P}(K) + \mathrm{P}(C \mid K^c)\,\mathrm{P}(K^c)}.$$

   Plugging in the values we get $\mathrm{P}(K \mid C) = 2/3$.

4. An unknown number of independent trials is performed, each of which succeeds with the same probability. You can only observe the number of successful trials. After many runs of this experiment you conclude that the expected number of successful trials is 6, and the variance of this number is 2. How many trials are performed?

   Solution: Let X denote the number of successful trials. Since each trial is performed independently and succeeds with the same probability. X is Binomial(n, p). We have $\mathrm{E}[X] = np = 6$ and $\mathrm{Var}[X] = np(1-p) = 2$. Solving the two equations gives us $p = \frac{2}{3}$ and $n = 9$.

5. 15 runners divided into 3 teams are to participate in a race. If a runner wins, everyone on his team gets a \$1 prize. Your objective is to minimize the expected amount of prize money given away. How do you divide the runners into teams? Assume each runner is equally likely to win.

   Solution: Suppose the three teams have size n_1, n_2, n_3. The sample space consists of the runners under equally likely outcomes. (The outcome of the experiment is the winner of the race.) Let X

be the amount of money given away. Then $X = X_1 + \cdots + X_{15}$ where X_i is the amount given to runner i. By linearity of expectation

$$\mathrm{E}[X] = \mathrm{E}[X_1] + \cdots + \mathrm{E}[X_{15}] = \mathrm{P}(X_1 = 1) + \cdots + \mathrm{P}(X_{15} = 1)$$

The event $X_i = 1$ happens when someone on runner i's team wins. Since there are n_1 runners with probability $n_1/15$ of winning, another n_2 runners with probability $n_2/15$ of winning, and so on, we get

$$\mathrm{E}[X] = n_1 \cdot n_1/15 + n_2 \cdot n_2/15 + n_3 \cdot n_3/15 = (n_1^2 + n_2^2 + n_3^2)/15.$$

This expression is minimised when $n_1 = n_2 = n_3 = 5$. You can see this by trying all possibilities. If this is too much work see Solution 2.

Alternative Solution: As in Solution 1 let X be the amount of money given away and let N be the number of people on a random *team*, where each team is chosen equally likely. Then

$$\begin{aligned}
\mathrm{P}(X = n) &= \frac{\text{number of runners on } n \text{ person teams}}{15} \\
&= \frac{n \cdot \text{number of } n \text{ person teams}}{15} \\
&= \frac{n \cdot \text{number of } n \text{ person teams/number of teams}}{15/3} \\
&= \tfrac{1}{5} \cdot n \cdot \mathrm{P}(N = n).
\end{aligned}$$

Then $\mathrm{E}[X] = \sum 5n^2\, \mathrm{P}(N = n) = \frac{1}{5}\,\mathrm{E}[N^2]$. Since $\mathrm{E}[N] = 5$, $\mathrm{E}[X]$ is minimised when $\mathrm{Var}[N] = \mathrm{E}[N^2] - \mathrm{E}[N]^2$ equals zero, which happens when $N = 5$ with probability 1 – namely every team has five runners on it.

Practice Midterm 2

1. 3 red balls and 3 blue balls are randomly arranged on a line. Let X be the position of the first blue ball. (E.g. for the arrangement `RBRBBR`, $X = 2$.) Find the probability mass function of X.

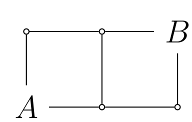
   Solution: The sample space consists of all arrangements of 3 red balls and 3 blue balls. We assume equally likely outcomes. The random variable X takes integer values from 1 to 4. X takes value x when the first $x - 1$ balls are red and the x-th ball is blue; the remaining $6 - x$ balls must then contain exactly two blue balls. Using the equally likely formula, we get that

   $$P(X = x) = \frac{\binom{6-x}{2}}{\binom{6}{3}} = \frac{(6-x)(5-x)}{40}$$

   or, in tabular form,

x	1	2	3	4
$P(X = x)$	1/2	3/10	3/20	1/20

2. Computers A and B are linked through seven cables as in the picture. Each cable fails with probability 10% independently of the others. What is the probability there is a connection between A and B?



Solution: There are four possible connections from A to B: (1) up-right-right, (2) right-right-up, (3) right-up-right, and (4) up-right-down-right-up. For connection (i), let E_i be the event that all cables on it are working. We are interested in the probability of the event $E = E_1 \cup E_2 \cup E_3 \cup E_4$. Using the inclusion-exclusion formula, we can express $P(E_1 \cup E_2 \cup E_3 \cup E_4)$ in terms of the probabilities of the intersections of various subsets of events among E_1, E_2, E_3, E_4. By independence, the probability of any such intersection of events is of the form 0.9^c, where c is the number of cables that the intersection depends on. Carrying out this plan we obtain

$$\begin{aligned} \mathrm{P}(E) &= \sum \mathrm{P}(E_i) - \sum \mathrm{P}(E_i \cap E_j) + \sum \mathrm{P}(E_i \cap E_j \cap E_k) - \mathrm{P}(E_1 \cap E_2 \cap E_3 \cap E_4) \\ &= (3 \times 0.9^3 + 0.9^5) - (2 \times 0.9^5 + 3 \times 0.9^6 + 0.9^7) + (4 \times 0.9^7) - (0.9^7) \\ &= 3 \times 0.9^3 - 0.9^5 - 3 \times 0.9^6 + 2 \times 0.9^7 \\ &\approx 0.959. \end{aligned}$$

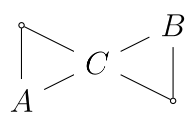
Alternative Solution: Let M be the event that the vertical cable in the middle fails. By the total expectation theorem,

$$P(E) = 0.1 \times P(E \mid M) + 0.9 \times P(E \mid M^c). \tag{1}$$

Conditioned on M, E happens when E_1 happens or E_2 happens (using the notation from Solution 1). Since E_1 and E_2 are independent of M and of one another,

$$P(E \mid M) = P(E_1 \cup E_2 \mid M) = P(E_1 \cup E_2) = 1 - P(E_1^c E_2^c) = 1 - P(E_1^c)P(E_2^c) = 1 - (1 - 0.9^3)^2.$$

Conditioned on M^c, we can contract the two vertices in the middle into a single vertex C (as there is an operational connection between them) and reduce the network to the following one:



Let F_1 be the event of a connection from A to C and F_2 the event of a connection from C to B. Then F_1 and F_2 are independent and so

$$P(E \mid M^c) = P(F_1 F_2) = P(F_1)P(F_2) = (1 - P(F_1^c))(1 - P(F_2^c)).$$

Now F_1^c happens when both of the connection from A to M fail. Since they are independent,

$$P(F_1^c) = 0.1 \times (1 - 0.9^2).$$

By symmetry, $P(F_2^c) = 0.1 \times (1 - 0.9^2)$. Plugging everything into (1), we get that

$$P(E) = 0.1 \times (1 - (1 - 0.9^3)^2) + 0.9 \times (1 - 0.1 \times (1 - 0.9^2))^2 \approx 0.959.$$

3. Toss a coin 4 times. Let X, Y and Z be the number of heads among the first two, middle two, and last two tosses, respectively. Are X and Z independent given that $Y \neq 1$? Justify carefully.

Solution: No. Given that $Y \neq 1$, the events $Y = 0$ and $Y = 2$ (two tails and two heads in the middle each happen with probability) half. By the total probability theorem,

$$\mathrm{P}(X = 0 | Y \neq 1) = \frac{1}{2}\mathrm{P}(X = 0 | Y = 0) + \frac{1}{2}\mathrm{P}(X = 0 | Y = 2) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 0 = \frac{1}{4}.$$

By symmetry, $\mathrm{P}(Z = 0 | Y \neq 1)$ is also $1/4$. However,

$$\begin{aligned}\mathrm{P}(X = 0, Z = 0 | Y \neq 1) &= \frac{1}{2}\mathrm{P}(X = 0, Z = 0 | Y = 0) + \frac{1}{2}\mathrm{P}(X = 0, Z = 0 | Y = 2) \\ &= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot 0 \\ &= \frac{1}{8},\end{aligned}$$

so the events $X = 0$ and $Z = 0$ are not independent conditioned on $Y = 1$. Therefore X and Z are not conditionally independent.

4. The average lifetime of a lightbulb is 10 months. You install 10 lightbulbs today. What is the probability that at least one of them fails within a month? Assume their failures are independent.

   Solution: To come up with a model for lightbulb failure, we partition each month into intervals of length $1/n$ (in months) for some large n. For every such interval, given that the lightbulb has not failed in the previous intervals, a failure happens with probability p. Since on average, we expect one failure every 10 months, $10np = 1$, so $p = 1/10n$. As n becomes large, we can describe the number of failures of a single lightbulb within a month as a Poisson(1/10) random variable.

   The probability that none of the lightbulbs fail within a month is then the probability that 10 independent Poisson(1/10) random variables are all zero, which equals $((1/10)^0 e^{-1/10}/0!)^{10} = 1/e$. So the probability at least one fails is $1 - 1/e \approx 0.632$.

5. Eight people's hats are mixed up and randomly redistributed. What is the expected number of pairs that exchanged hats (Alice got Bob's and Bob got Alice's)?

   Solution: For every two people i and j we introduce a random variable X_{ij} that takes value 1 if the two exchanged hats and 0 if not. The expected value of X_{ij} is the probability that i and j exchanged hats, which is $1/8 \cdot 7$: Bob gets Alice's hat with probability $1/8$, and given this happened Alice gets Bob's with probability $1/7$. The number of pairs that exchanged hats is $X_{12} + X_{13} + \cdots + X_{78}$, where the indices range over all (ordered) $\binom{8}{2}$ pairs of people. By linearity of expectation,

$$\mathrm{E}[X_{12} + X_{13} + \cdots + X_{78}] = \mathrm{E}[X_{12}] + \mathrm{E}[X_{13}] + \cdots + \mathrm{E}[X_{78}] = \binom{8}{2} \cdot \frac{1}{8 \cdot 7} = \frac{1}{2}.$$