Practice Final 1

1. Urn A has 4 blue balls. Urn B has 1 blue ball and 3 red balls.

   (a) You draw a ball from a random urn and it is blue. What is the probability that it came from urn A?

   Solution: Let B_1 be the event the ball is blue and A be the event the ball came from urn A. By Bayes' rule

$$\mathrm{P}(A|B_1) = \frac{\mathrm{P}(B_1|A)\,\mathrm{P}(A)}{\mathrm{P}(B_1|A)\,\mathrm{P}(A) + \mathrm{P}(B_1|A^c)\,\mathrm{P}(A^c)} = \frac{1 \cdot (1/2)}{1 \cdot (1/2) + (1/4) \cdot (1/2)} = \frac{4}{5}.$$

   (b) You draw another ball from the same urn. What is the probability that the second ball is also blue?

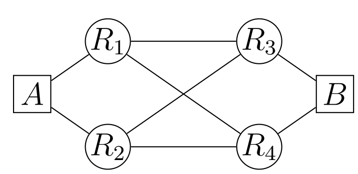
   Solution: Let B_2 be the event that the second ball is blue. By the total probability theorem and Bayes' rule

$$\mathrm{P}(B_2|B_1) = \frac{\mathrm{P}(B_2 \cap B_1)}{\mathrm{P}(B_1)} = \frac{\mathrm{P}(B_2 \cap B_1|A)\,\mathrm{P}(A) + \mathrm{P}(B_2 \cap B_1|A^c)\,\mathrm{P}(A^c)}{\mathrm{P}(B_1|A)\,\mathrm{P}(A) + \mathrm{P}(B_1|A^c)\,\mathrm{P}(A^c)}$$
$$= \frac{1 \cdot (1/2) + (1/4)^2 \cdot (1/2)}{1 \cdot (1/2) + (1/4) \cdot (1/2)} = \frac{17}{20}.$$

2. Computers A and B are linked through routers R_1 to R_4 as in the picture. Each router fails independently with probability 10%.

   (a) What is the probability there is a connection between A and B?

   Solution: Let R_i be the event that router i is operational. The event "there is a connection between A and B" is $(R_1 \cup R_2) \cap (R_3 \cup R_4)$. By independence



$$\begin{aligned}\mathrm{P}((R_1 \cup R_2) \cap (R_3 \cup R_4)) &= \mathrm{P}(R_1 \cup R_2)\,\mathrm{P}(R_3 \cup R_4) \\ &= (1 - \mathrm{P}(R_1^c \cap R_2^c))(1 - \mathrm{P}(R_1^c \cap R_2^c)) \\ &= (1 - \mathrm{P}(R_1^c)\,\mathrm{P}(R_2^c))(1 - \mathrm{P}(R_3^c)\,\mathrm{P}(R_4^c)) \\ &= (1 - 0.1^2)^2 \\ &= 0.9801.\end{aligned}$$

   (b) Are the events "there is a connection between A and B" and "exactly two routers fail" independent? Justify your answer.

Solution: No. The probability that there is a connection between A And B given that exactly two routers fail is 2/3: Given that exactly two routers fail, the failed routers are equally likely to be any of the 6 pairs $R_1R_3, R_1R_4, R_2R_3, R_2R_4, R_1R_2, R_3R_4$, and there is a connection between A and B in the first 4 out of these 6 possibilities. This probability is not equal to the unconditional probability from part (a) and so the two events are not independent.

3. A bus takes you from A to B in 10 minutes. On average a bus comes once every 5 minutes. A taxi takes you in 5 minutes, and on average a taxi comes once every 10 minutes. Their arrival times are independent exponential random variables. A bus comes first.

   (a) If you want to minimize the (expected) travel time, should you take this bus?

   Solution: Yes. If you waited for a taxi your expected travel time would be the expected waiting time for the next taxi which is 10 minutes plus its travel time which is another 5 minutes for a total of 15 minutes.

   (b) If you do take the bus, what is the probability that you made the wrong decision?

   Solution: The probability of a wrong decision is the probability that a taxi arrives within the next five minutes, which is the probability that an Exponential(1/10) random variable is less than 5, which is $1 - e^{-5/10} = 1 - e^{-1/2} \approx 39.35\%$.

4. 10 people toss their hats and each person randomly picks one. The experiment is repeated one more time.

   (a) What is the probability that Bob picked his own hat both times?

   Solution: By independence, the probability that Bob picked his hat both times is the product of the probabilities that he picked it in each trial, so it is $(1/10) \cdot (1/10) = 1/100$.

   (b) Let A be the event that at least one person picked their own hat both times. True or false: $P(A) > 25\%$? Justify your answer.

   Solution: False. Let X_i take value 1 if person i picked their hat both times. A occurs if $X = X_1 + \cdots + X_{10} \geq 0$. By part (a) and linearity of expectation, $\mathrm{E}[X] = 10 \cdot (1/100) = 0.1$. By Markov's inequality, $\mathrm{P}[X \geq 1] \leq \mathrm{E}[X]/1 = 0.1$ which is less than 25%.

5. X is a Normal$(0, \Theta)$ random variable, where the prior PMF of the parameter Θ is $\mathrm{P}(\Theta = 1/2) = 1/2$, $\mathrm{P}(\Theta = 1) = 1/2$. You observe the following three independent samples of X: $1.0, 1.0, -1.0$.

   (a) What is the posterior PMF of Θ?

   Solution: By Bayes' rule

$$f_{\Theta|X_1X_2X_3}(\theta|1.0, 1.0, -1.0) \propto f_{X_1X_2X_3|\Theta}(1.0, 1.0, -1.0|\theta)\,\mathrm{P}(\Theta = \theta) \propto \frac{1}{\theta^3} e^{-3/2\theta^2}\,\mathrm{P}(\Theta = \theta).$$

   As Θ is equally likely to take values 0 and 1/2, the posterior PMF is

$$f_{\Theta|X_1X_2X_3}(1/2|1.0, 1.0, -1.0) = \frac{8e^{-6}}{8e^{-6} + e^{-3/2}} \quad f_{\Theta|X_1X_2X_3}(1|1.0, 1.0, -1.0) = \frac{e^{-3/2}}{8e^{-6} + e^{-3/2}}.$$

(b) What is the MAP estimate of Θ?

Solution: As $e^{-3/2} \approx 0.2231$ is larger than $8e^{-6} \approx 0.0198$ the MAP estimate is $\hat{\Theta} = 1$.

(c) What is the posterior probability that $|X| \geq 1$?

Solution: The posterior probabilities of Θ are 1/2 with probability about $0.0198/(0.2231 + 0.0198) \approx 0.0815$ and 1 with probability about $0.2231/(0.2231 + 0.0198) \approx 0.9185$. By the total probability theorem the posterior probability of $|X| \geq 1$ is about

$$\begin{aligned} 0.0185 \cdot \mathrm{P}(|\mathrm{Normal}(0, 1/2)| \geq 1) &+ 0.9185\, \mathrm{P}(|\mathrm{Normal}(0, 1)| \geq 1) \\ &\approx 0.0185 \cdot 2\, \mathrm{P}(\mathrm{Normal}(0, 1) \geq 2) + 0.9185 \cdot 2\, \mathrm{P}(\mathrm{Normal}(0, 1) \geq 1) \\ &\approx 0.0185 \cdot 2 \cdot 0.023 + 0.9185 \cdot 2 \cdot 0.159 \\ &\approx 0.2929. \end{aligned}$$

Practice Final 2

1. Let X, Y, Z be independent Binomial$(2, \frac{1}{2})$ random variables.

(a) What is the conditional PMF of X conditioned on $X \neq Z$?

Solution: The joint PMF is

$$\begin{aligned} \mathrm{P}(X = 0, X \neq Z) = \mathrm{P}(X = 0, Z = 1) + \mathrm{P}(X = 0, Z = 2) &= \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{4} = \frac{3}{16} \\ \mathrm{P}(X = 1, X \neq Z) = \mathrm{P}(X = 1, Z = 0) + \mathrm{P}(X = 1, Z = 2) &= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} = \frac{4}{16} \\ \mathrm{P}(X = 2, X \neq Z) = \mathrm{P}(X = 2, Z = 0) + \mathrm{P}(X = 2, Z = 1) &= \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{2} = \frac{3}{16}. \end{aligned}$$

The conditional PMF is the joint PMF normalized by $\mathrm{P}(X \neq Z)$, which is

$$\mathrm{P}(X = 0 | X \neq Z) = \frac{3}{10}, \quad \mathrm{P}(X = 1 | X \neq Z) = \frac{4}{10}, \quad \mathrm{P}(X = 2 | X \neq Z) = \frac{3}{10}.$$

(b) Are X and Y independent conditioned on $(X \neq Z)$ AND $(Y \neq Z)$?

Solution: No. We show that

$$\mathrm{P}(X = 1 \mid X \neq Z, Y \neq Z)\, \mathrm{P}(Y = 1 \mid X \neq Z, Y \neq Z) \neq \mathrm{P}(X = 1, Y = 1 \mid X \neq Z, Y \neq Z).$$

By symmetry the two probabilities on the left are the same. We calculate these expressions:

$$\begin{aligned}
\mathrm{P}(X \neq Z, Y \neq Z) &= \sum_{z \in \{0,1,2\}} \mathrm{P}(X \neq Z, Y \neq Z | Z = z)\,\mathrm{P}(Z = z) \\
&= (3/4)^2 \cdot (1/4) + (1/2)^2 \cdot (1/2) + (3/4)^2 \cdot (1/4) \\
&= 13/32, \\
\mathrm{P}(X = 1, X \neq Z, Y \neq Z) &= \mathrm{P}(X = 1, Z \neq 1, Y \neq Z) \\
&= \mathrm{P}(X = 1)\,\mathrm{P}(Z \neq 1, Y \neq Z) \\
&= (1/2) \cdot (\mathrm{P}(Y \neq 0, Z = 0) + \mathrm{P}(Y \neq 2, Z = 2)) \\
&= (1/2) \cdot 2 \cdot (1/4) \cdot (3/4) \\
&= 3/16, \\
\mathrm{P}(X = 1, Y = 1, X \neq Z, Y \neq Z) &= \mathrm{P}(X = 1, Y = 1, Z \neq 1) \\
&= (1/2)(1/2)(1/2) \\
&= 1/8.
\end{aligned}$$

By the conditional probability formula the expression on the left is $((3/16)/(13/32))^2 \approx 0.2130$ and the one on the right is $(1/8)/(13/32) \approx 0.3077$. These are not equal.

2. Alice and Bob decide to meet somewhere. Alice's arrival time A is uniform between 12:00 and 12:45. Bob's arrival time B is uniform between 12:15 and 1:00. Their arrival times are independent.

   (a) Let f_{A-B} be the PDF of $A - B$. What is $f_{A-B}(0)$?

   Solution: We model A and B as Uniform$(0, 3/4)$ and Uniform$(1/4, 1)$ random variables respectively (at the hour scale). By the convolution formula, $f_{A-B}(0) = \int_{-\infty}^{\infty} f_A(t) f_B(t) dt$, where f_A, f_B are the PDFs of A and B. $f_A(t) f_B(t)$ takes value $(4/3)^2$ when t is between $1/4$ and $3/4$ and 0 otherwise, so the integral equals $(1/2) \cdot (4/3)^2 = 8/9$. (If time is scaled in minutes the answer is 60 times smaller.)

   (b) What is the probability that Bob arrives before Alice?

   Solution: The event that Bob arrives before Alice is the value of the integral $\int_{a>b} f_A(a) f_B(b) da db$. The value of the integrand is $(4/3)^2$ when (a, b) is in the interior of the triangle with vertices $(1/4, 1/4), (1/4, 3/4), (3/4, 3/4)$ and zero elsewhere. The triangle has area $(1/2)^2/2 = 1/8$. Therefore $\mathrm{P}(A > B) = (1/8)(4/3)^2 = 2/9$.

3. Let $Y = AX + B$ where A, B, X are independent Normal$(0, 1)$ random variables.

   (a) What is $\mathrm{Var}[\mathrm{E}[Y|X]]$?

   Solution: By linearity of expectation, $\mathrm{E}[AX+B|X] = \mathrm{E}[A]X+\mathrm{E}[B] = 0$ so $\mathrm{Var}[\mathrm{E}[Y|X]] = 0$.

   (b) What is $\mathrm{E}[\mathrm{Var}[Y|X]]$?

   Solution: By independence, $\mathrm{Var}[AX+B|X] = \mathrm{Var}[AX|X]+\mathrm{Var}[B] = X^2\,\mathrm{Var}[A]+\mathrm{Var}[B] = X^2 + 1$, so $\mathrm{E}[\mathrm{Var}[Y|X]] = \mathrm{E}[X^2 + 1] = \mathrm{Var}[X] + 1 = 2$.

4. Boys and girls arrive independently at a meeting point at a rate of one boy per minute and one girl per minute, respectively. Let T be the first time at which both a boy and a girl have arrived.

(a) Find the cumulative distribution function (CDF) of T.

Solution: The probability that a boy has arrived by time t is $1 - e^{-t}$, i.e. the CDF of an Exponential(1) random variable. The probability that a boy has arrived by time t is therefore $1 - e^{-t}$, and same for a girl. The events are independent, the probability that both have arrived by time t is $\mathrm{P}(T \leq t) = (1 - e^{-t})^2$ if $t \geq 0$ and 0 if not.

(b) What is the expected value of T? (**Hint:** You don't *have* to use calculus.)

Solution: We can write $T = T_1 + T_2$ where T_1 is the arrival time of the first person and T_2 is the arrival time of the next person *of the opposite gender*. As people arrive at a rate of two per minute, T_1 is an Exponential(2) random variable. By the memoryless property T_2 is an Exponential(1) random variable. Therefore $\mathrm{E}[T] = \mathrm{E}[T_1] + \mathrm{E}[T_2] = 1/2 + 1 = 3/2$.

5. A deck of cards is divided into 26 pairs. Let X be the number of those pairs in which both cards are of the same suit. (A deck of cards has 4 suits and each suit has 13 cards.)

(a) What is the expected value of X?

Solution: We can write $X = X_1 + \cdots + X_{26}$ where X_i is 1 if the cards in the i-th pair are of the same suit and 0 if not. Then $\mathrm{E}[X_i] = \mathrm{P}(X_i = 1)$ is the probability that the i-th pair's cards are of the same suit, which is 12/51 because conditioned on the first card's suit, there are 12 out of 51 identical choices for the second one. By linearity of expectation $\mathrm{E}[X] = \mathrm{E}[X_1] + \cdots + \mathrm{E}[X_{26}] = 26 \cdot 12/51 \approx 6.118$.

(b) What is the variance of X?

Solution: The variance of X is the sum of the 26 variances of X_i and the $26 \cdot 25$ covariances of X_i and X_j. The variance of X_i is $v = (12/51) \cdot (1 - 12/51) \approx 0.1799$. The covariance of X_i and X_j is

$$\mathrm{E}[X_i X_j] - \mathrm{E}[X_i]\,\mathrm{E}[X_j] = \mathrm{P}(X_i = 1, X_j = 1) - \mathrm{P}(X_i = 1)\,\mathrm{P}(X_j = 1).$$

The term $\mathrm{P}(X_i = 1, X_j = 1)$ is the probability of the event A that within both the i-th pair and the j-th pair, both cards are of the same suit. We can calculate this using the total probability theorem applied to the event E that the first card of the i-th pair and the first card of the j-th pair are of the same suit:

$$\mathrm{P}(X_i = 1, X_j = 1) = \mathrm{P}(A) = \mathrm{P}(A|E)\,\mathrm{P}(E) + \mathrm{P}(A|E^c)\,\mathrm{P}(E^c).$$

The probability of E is 12/51. Conditioned on E, A happens if the second cards of both pairs are also of the same suit, which is $11/50 \cdot 10/49$. Conditioned on E^c—for example, if the i-th pair's first card is a heart and the j-th pair first card is a spade—A happens if the second cards are a heart and a spade respectively, which happens with probability $(12/50) \cdot (12/49)$, and so

$$\mathrm{P}(A) = \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{12}{51} + \frac{12}{50} \cdot \frac{12}{49} \cdot \left(1 - \frac{12}{51}\right).$$

Therefore the covariance of X_i and X_j equals

$$c = \mathrm{P}(A) - \left(\frac{12}{51}\right)^2 \approx 0.0001469.$$

Finally, $\mathrm{Var}[X] = 26 \cdot v + 26 \cdot 25 \cdot c \approx 4.7737$.