1. Let N be the number of times you flip a fair coin until you observe both a head and a tail. For example if the outcome is `HHHT` then $N = 4$. What is $\mathrm{E}[N]$ and $\mathrm{Var}[N]$?

   Solution: $N = 1 + G$ where G is a Geometric(1/2) random variable: If the first toss is head you keep tossing a fair coin until you see a tail, and vice versa. Therefore $\mathrm{E}[N] = 1 + \mathrm{E}[G] = 3$ and $\mathrm{Var}[N] = \mathrm{Var}[G] = (1 - 1/2)/(1/2)^2 = 2$.

2. You flip a p-biased coin (heads with probability p) 10 times. For which value(s) of p, if any, are the events $A =$ "the first two flips are heads" and $B =$ "there are exactly two heads" independent?

   Solution: . Event A has probability p^2. Event B has probability $\binom{10}{2}p^2(1-p)^8$. Event $A \cap B$ occurs when two heads are followed by eight tails, so it has probability $p^2(1-p)^8$. So A and B are independent if and only if

$$p^2 \cdot \binom{10}{2} p^2(1-p)^8 = p^2(1-p)^8.$$

   This equation has four solutions: $p = 0$, $p = 1$ and $p = \pm\binom{10}{2}^{-1/2} = \pm 1/\sqrt{45}$. Discarding the negative one, the desired values of p are 0, 1, and $1/\sqrt{45}$.

3. Keep rolling a 3-sided die until the sum of the values strictly exceeds 2. For example if the first roll is a 1 then you roll again; if the second roll is then a 2, $1 + 2 = 3 > 2$ and you stop. Find the PMF of the number of times you rolled.

   Solution: The random variable X of interest can take values 1, 2, and 3. $X = 1$ happens if and only if the first roll is a 3, so $\mathrm{P}(X = 1) = 1/3$. $X = 3$ happens if and only if the first two rolls are both 1, so $\mathrm{P}(X = 3) = 1/9$. Since probabilities must add up to 1, we get that $\mathrm{P}(X = 2) = 1 - 1/3 - 1/9 = 5/9$.

4. On a light rain day, rain falls at an average rate of 1 drop per second. On a heavy rain day, the average rate is 2 drops per second. 2/3 of the rainy days are light and 1/3 are heavy. You walk out and 2 drops of rain hit you in the next second. What is the probability it is a light rain day?

   Solution: Let L and H be the events of light and heavy rain and N be the number of drops that hit you. We model $N \mid L$ and $N \mid H$ as Poisson(1) and Poisson(2) random variables, respectively. By Bayes' rule,

$$\begin{aligned}
\mathrm{P}(L|N=2) &= \frac{\mathrm{P}(N=2|L)\,\mathrm{P}(L)}{\mathrm{P}(N=2|L)\,\mathrm{P}(L) + \mathrm{P}(N=2|H)\,\mathrm{P}(H)} \\
&= \frac{(1^2e^{-1}/2)\cdot(2/3)}{(1^2e^{-1}/2)\cdot(2/3) + (2^2e^{-2}/2)\cdot(1/3)} \\
&= \frac{e}{e+2} \approx 0.576.
\end{aligned}$$

5. Let N be the number of distinct values observed when a 6-sided die is rolled 6 times. For example, if the outcome is `521154` then the observed values are $\{\mathtt{1}, \mathtt{2}, \mathtt{4}, \mathtt{5}\}$ and $N = 4$. What is $\mathrm{E}[N]$?

   Solution: We can write $N = N_1 + N_2 + N_3 + N_4 + N_5 + N_6$ where N_1 takes value 1 if a `1` was observed and 0 if not, and similarly for the others. Then $\mathrm{E}[N_1] = \mathrm{P}[N_1 = 1]$ is the probability that a `1` was observed. This is one minus the probability that a `1` was not observed, which is $(5/6)^6$ by independence. Summarizing, $\mathrm{E}[N_1] = 1 - (5/6)^6$. By symmetry, $\mathrm{E}[N_1] = \cdots = \mathrm{E}[N_6]$, so $\mathrm{E}[N] = 6(1 - (5/6)^6) \approx 3.991$.