Practice questions

1. Alice, Bob, and Charlie are equally likely to have been born on any three days of the year. Let E_{AB} be the event that Alice and Bob were born on the same day. Define E_{BC} and E_{CA} analogously. Which of the following statements is true:

   (a) Any two of the three events E_{AB}, E_{BC}, E_{CA} are independent.

   (b) E_{AB}, E_{BC}, and E_{CA} are independent.

   (c) E_{AB} and E_{BC} are independent conditioned on E_{CA}.

2. You go to a party with 500 guests. What is the probability that exactly one other guest has the same birthday as you? Calculate this exactly and also approximately by using the Poisson PMF $p(k) = \lambda^k e^{-\lambda}/k!$. (For simplicity, exclude birthdays on February 29.) *(Textbook problem 2.2)*

3. Let X be the number of 1's you observed after rolling a fair dice for n times. Show that PMF of X can be computed by starting with $p_X(0) = (1-\frac{1}{6})^n$ and then using the recursive formula

$$p_X(k+1) = \frac{1}{5} \cdot \frac{n-k}{k+1} \cdot p_X(k), \quad k = 0, 1...., n-1.$$

   (Adapted from Textbook problem 2.8)

4. An ENGG 2430A tutorial meets for 11 weeks. Each week, the TA asks 5 questions and chooses 5 random but distinct students to answer them, independently of what happened in previous weeks. If you are one of 40 students in the tutorial (and attendance is always perfect!), what is the probability that you are chosen in the final week but not before that?

5. You are given a biased coin with probability p of getting head and you toss the coin for X times until you first see a head. Find the PMF of the random variable $Y = X \bmod 3$. For example, if the sequence is `TTTH`, then $X = 4$ and $Y = 4 \bmod 3 = 1$.

Additional ESTR 2002 questions

6. Can there be four events E_1, E_2, E_3, E_4 so that every pair E_i, E_j is independent but every triple E_i, E_j, E_k is not (i, j, k are distinct indices)?

7. Consider a model with 4 events G, V, R, and S, meaning *gray*, *Vancouver*, *rain*, and *sad*, respectively. The directed graphical model describing their relationship $V \to R \leftarrow G \to S$, and the conditional probabilities are

$$\mathrm{P}(R|V^c \cap G^c) = 0.4 \;\; \mathrm{P}(R|V^c \cap G) = 0.7 \;\; \mathrm{P}(R|V \cap G^c) = 0.8 \;\; \mathrm{P}(R|V \cap G) = 0.9$$

   (a) Let $\mathrm{P}(V^c) = \delta$, $\mathrm{P}(G^c) = \alpha$, $\mathrm{P}(S^c|G^c) = \gamma$, $\mathrm{P}(S^c|G) = \beta$. Write $\mathrm{P}(S|V)$ in terms of $\alpha, \beta, \gamma, \delta$;

   (b) Do the same for $\mathrm{P}(S|V^c)$. Is this the same as $\mathrm{P}(S|V)$? Explain why.

   (c) Find the values of $\alpha, \beta, \gamma, \delta$ for which each of the following three instances have the maximum probability: $V \cap G \cap R \cap S$, $V \cap G \cap R^c \cap S$, $V \cap G^c \cap R^c \cap S^c$.