Questions

1. A Chicken lays a Poisson(λ) number N of eggs. Each egg independently hatches a chick with probability p. Let X be the number of chicks that hatch. Calculate

   (a) the conditional expectation $\mathrm{E}[X|N=n]$;

   Solution: X is a Binomial(N,p) random variable, so $\mathrm{E}[X|N=n]=np$.

   (b) the unconditional expectation $\mathrm{E}[X]$;

   Solution: By the total expectation theorem,

$$\mathrm{E}[X]=\sum_{n=0}^{\infty}\mathrm{E}[X|N=n]\,\mathrm{P}(N=n)=np\cdot\mathrm{P}(N=n)=\mathrm{E}[N]p=\lambda p.$$

   (c) the unconditional expectation $\mathrm{E}[NX]$;

   Solution: By the total expectation theorem again,

$$\begin{aligned}
\mathrm{E}[NX]&=\sum_{n=0}^{\infty}\mathrm{E}[NX|N=n]\,\mathrm{P}(N=n)\\
&=\sum_{n=0}^{\infty}n\cdot\mathrm{E}[X|N=n]\cdot\mathrm{P}(N=n)\\
&=\sum_{n=0}^{\infty}n\cdot np\cdot\mathrm{P}(N=n)\\
&=\sum_{n=0}^{\infty}n^2\,\mathrm{P}(N=n)\cdot p\\
&=\mathrm{E}[N^2]p\\
&=(\mathrm{Var}[N]+\mathrm{E}[N]^2)p\\
&=(\lambda+\lambda^2)p.
\end{aligned}$$

   (d) the covariance $\mathrm{Cov}[X,N]$.

   Solution: $\mathrm{Cov}[X,N]=\mathrm{E}[NX]-\mathrm{E}[N]\,\mathrm{E}[X]=(\lambda+\lambda^2)p-\lambda\cdot\lambda p=\lambda p$.

2. You draw 10 balls at random among 15 red and 5 blue balls. Let X be the number of red balls drawn.

   (a) What is the expected value of X?

   Solution: Let $X=X_1+X_2+\cdots+X_{10}$, where X_i indicates if the i-th drawn ball is red. By linearity of expectation,

$$\mathrm{E}[X]=\mathrm{E}[X_1]+\cdots+\mathrm{E}[X_{10}]=10\cdot\frac{15}{20}=7.5.$$

   (b) Write $X=X_1+X_2+\cdots+X_{10}$, where X_i indicates if the i-th drawn ball is red. What is the variance of X_i?

   Solution: $\mathrm{Var}[X_i]=\mathrm{E}[X_i^2]-\mathrm{E}[X_i]^2=\mathrm{P}(X_i=1)-\mathrm{P}(X_i=1)^2=\frac{3}{4}-(\frac{3}{4})^2=\frac{3}{16}$.

(c) What is the covariance of X_i and X_j ($i \neq j$)?

Solution: $\mathrm{Cov}[X_i, X_j] = \mathrm{E}[X_i X_j] - \mathrm{E}[X_i]\,\mathrm{E}[X_j] = \mathrm{P}(X_i = 1, X_j = 1) - \mathrm{P}(X_i = 1)\,\mathrm{P}(X_j = 1) = \frac{15}{20} \cdot \frac{14}{19} - (\frac{15}{20})^2 = -\frac{3}{304}$. The variables X_i and X_j are negatively correlated: Given that ball i is red, ball j is less likely to be red.

(d) What is the variance of X?

Solution: The variance of X is the sum of the 10 variances from part (b) plus the $10 \cdot 9$ covariances from part (c), so $\mathrm{Var}[X] = 10 \cdot \frac{3}{16} + 10 \cdot 9 \cdot \frac{-3}{304} = 0.9868$.

3. Two typing monkeys sit at special keyboards. The keyboards have only two letters, `a` and `b`. Each monkey types in a random 200 letter string, independently of the other one. Let E be the event "There is a pattern of 20 consecutive letters that appears in both strings." Show that $\mathrm{P}(E) < 5\%$.

Solution: Let's call the two monkeys Alice and Bob and let X be the number of identical 20-letter patterns typed by the two monkeys. We can write

$$X = \sum_{i=1}^{181} \sum_{j=1}^{181} X_{i,j}$$

where $X_{i,j}$ is an indicator random variable for the event that Alice's 20-letter pattern starting at position i equals Bob's 20-letter pattern starting at position j. By linearity of expectation

$$E[X] = \sum_{i=1}^{181} \sum_{j=1}^{181} E[X_{i,j}] = \sum_{i=1}^{181} \sum_{j=1}^{181} P(X_{i,j} = 1).$$

Let's start by calculating $P(X_{1,1} = 1)$. This is the probability that Alice's first 20 letters are identical to Bob's first 20 letters. The probability that their first letter is the same is $1/2$, and the same is true for the second letter, up to the 20-th letter. By independence, the probability that their first 20 letter patterns are identical is $1/2^{20}$. By the same reasoning, $P(X_{i,j} = 1) = 1/2^{20}$ for all i, j and so

$$E[X] = (181)^2/2^{20} \approx 0.035.$$

By Markov's inequality, $P(X \geq 1) \leq E[X] < 5\%$.

4. 100 people put their hats in a box and each one pulls a random hat out.

(a) Let G be any 10-person group. What is the probability that everyone in G pulls their own hat?

Solution: The probability that the first person in the group pulls their own hat is $1/100$. Given this happened, the probability that the second person in the group does so is $1/99$, and so on. So the probability that everyone in the group succeeds is $1/(100 \cdot 99 \cdots 91)$.

(b) What is the expected *number* of 10-person groups in which everyone pulls their own hat?

Solution: Let X_S be the random variable indicating that everyone in group S pulled their own hat. Then X is the sum of the random variables X_S. By linearity of expectation, $\mathrm{E}[X]$ is the sum of $\mathrm{E}[X_S] = 1/(100 \cdot 99 \cdots 91)$ over all 10-person groups S. As there are $\binom{100}{10}$ ways to choose a 10-person group,

$$\mathrm{E}[X] = \binom{100}{10} \cdot \frac{1}{100 \cdot 99 \cdots 91} = \frac{1}{10!}.$$

(c) Show that the probability that 10 or more people pull their own hat is less than 10^{-6}.

Solution: By Markov's inequality, the probability that at least one group succeeded in pulling all of their own hats is at most

$$P(X \geq 1) \leq \frac{E[X]}{1} = \frac{1}{10!} \approx 2.7557 \times 10^{-7} < 10^{-6}$$