1. The joint PDF of $X$ and $Y$ is

$$f_{X,Y}(x,y) = \begin{cases} C(x+y+1)y, & \text{if } 0 \le x \le 2, 0 \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Find (a) the value of $C$ and (b) The conditional PDF $f_{Y|X}(y|x)$.

**Solution:**

(a) The PDF must integrate to one, so

$$1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)dxdy = \int_0^2\int_0^2 C(x+y+1)ydxdy = \frac{40}{3}C.$$

Therefore $C = \frac{3}{40}$.

(b) $f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy = \int_0^2 C(x+y+1)ydy = C(2x+\frac{14}{3})$. Using the convolution formula,

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{(x+y+1)y}{2x+\frac{14}{3}}.$$

2. Alice and Bob agree to meet. Alice's arrival time $A$ is uniform between 12:00 and 12:45 and Bob's arrival time $B$ is uniform between 12:15 and 13:00. Let $E$ be the event "Alice and Bob arrive within 30 minutes of one another".

(a) What is $\mathrm{P}(E)$ assuming $A$ and $B$ are independent?

(b) If you don't know the joint PDF of $A$ and $B$, how large can $\mathrm{P}(E)$ be?

(c) **(Optional)** If you don't know the joint PDF of $A$ and $B$, how small can $\mathrm{P}(E)$ be?

**Solution:**

(a) We can model $A$ as a Uniform$(0, 45)$ random variable and $B$ as an independent Uniform$(15, 60)$ random variable. Their marginal PMFs are:

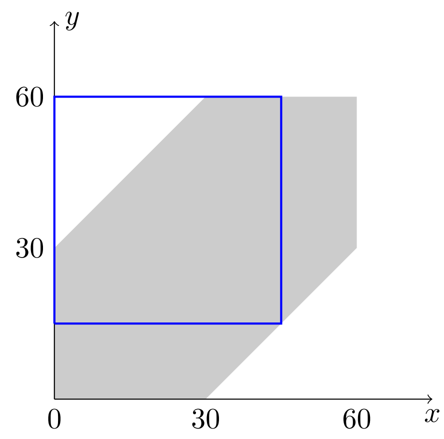
$$f_A(x) = \begin{cases} 1/45 & \text{if } 0 \le x < 45 \\ 0 & \text{otherwise} \end{cases}$$

$$f_B(y) = \begin{cases} 1/45 & \text{if } 15 \le y < 60 \\ 0 & \text{otherwise} \end{cases}$$

Because $A$ and $B$ are independent, their joint PMF is

$$f(x,y) = f_A(x)f_B(y) = \begin{cases} 1/2025 & \text{if } 0 \le x < 45 \text{ and } 15 \le y < 60 \\ 0 & \text{otherwise} \end{cases}$$

The event $E$ is $|B - A| < 30$. We want to calculate $\mathrm{P}(E) = \int_E f(x,y)dxdy$.




To understand this expression look at the above picture. The blue square consists of the points $(x, y)$ where $f(x, y)$ is nonzero (it equals $1/2025$). The grey strip are the points $x, y$ such that $|x - y| \le 30$. The intersection of the two consists of the blue square minus the triangle $T$ with vertices $(0, 30), (30, 30), (0, 60)$ so

$$\mathrm{P}(E) = \int_E f(x, y)dxdy = 1 - \int_T \frac{1}{2025}dxdy = 1 - \frac{\text{area}(T)}{2025} = 1 - \frac{1}{2025} \times 450 = \frac{7}{9}.$$

(b) If $B = A + 15$, then the marginal PMFs of $A$ and $B$ are as in part (a), but $\mathrm{P}(E) = \mathrm{P}(|B - A| < 30) = \mathrm{P}(15 < 30) = 1$.

(c) Let $E_A$ be the event $15 < A < 30$ and $E_B$ be the event $15 < B < 30$. By the axioms of probability,

$$\mathrm{P}(E) \ge \mathrm{P}(E_A \cap E_B) = \mathrm{P}(E_A) + \mathrm{P}(E_B) - \mathrm{P}(E_A \cup E_B) \ge \mathrm{P}(E_A) + \mathrm{P}(E_B) - 1.$$

As $\mathrm{P}(E_A^c) = \mathrm{P}(E_B^c) = 1/3$, we can conclude that $\mathrm{P}(E)$ must be at least $1/3$.
To argue that $\mathrm{P}(E)$ cannot in general be smaller we describe a probability model in which $\mathrm{P}(E) \le 1/3$. One way to do this is by specifying the marginal PDF of $B$ given $A$ as follows: $B = A + 30$ when $0 < A < 30$ and $B = 60 - A$ otherwise. It is easy to check that when $A$ is Uniform$(0, 45)$ then $B$ is Uniform$(0, 60)$, yet $\mathrm{P}(|B - A| < 30) = \mathrm{P}(A \ge 30) = 1/3$.

3. Raindrops hit the ground at a rate of 1 per second. An observatory has a raindrop sensing equipment. A signal is received by the computer with a maximum delay of 1 second after sensing a raindrop, with all delays equally likely. Find

(a) The joint PDF of the time $T$ of the first raindrop and the time $S$ of the signal reception.
(b) The marginal PDF of $S$.
(c) The conditional PDF of $T$ given $S$.

**Solution:**

(a) $S$ is a Uniform$(T, T+1)$ random variable, where $T$ is an Exponential(1) random variable. The joint PDF is $f_{S,T}(s, t) = f_T(t) f_{S|T}(s|t)$. We are given that

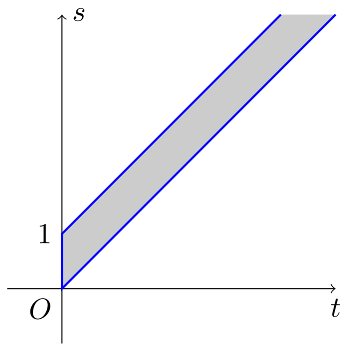
$$f_T(t) = \begin{cases} e^{-t} & \text{if } t \ge 0 \\ 0 & \text{otherwise} \end{cases}, \qquad f_{S|T}(s|t) = \begin{cases} 1 & \text{if } t \le s \le t+1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore $f_{S,T}(s, t) = f_T(t) f_{S|T}(s|t) = \begin{cases} e^{-t} & \text{if } t \ge 0,\ t \le s \le t+1 \\ 0 & \text{otherwise} \end{cases}$

*Alternative Solution:* We can write $S = T + D$ where the *delay* $D$ is a Uniform$(0,1)$ random variable that is independent of $T$. $S$ and $T$ take values $s$ and $t$ whenever $T$ and $D$ take values $t$ and $s - t$, respectively, so by independence

$$f_{S,T}(s,t) = f_T(t) f_D(s-t) = \begin{cases} e^{-t} & \text{if } t \geq 0,\ 0 \leq s-t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) The marginal PDF of $S$ is $f_S(s) = \int_0^{+\infty} f_{S,T}(s,t)dt$. Let $R$ be the region greyed out in the following plot in which $f_{S,T}(s,t)$ takes nonzero value $e^{-t}$.




Thus the integrand equals $e^{-t}$ with bounds 0 to $s$ when $s$ is between 0 and 1, and $s$ to $s+1$ when $s$ is larger than 1. This gives

$$f_S(s) = \int_0^s e^{-t} dt = 1 - e^{-s}, \qquad \text{when } 0 \leq s \leq 1,$$

$$f_S(s) = \int_{s-1}^s e^{-t} dt = e^{-s}(e-1), \qquad \text{when } s > 1.$$

*Alternative Solution:* As $T$ and $D$ are independent we can calculate the PDF of $S$ using the convolution formula:

If $s \geq 1$: $$f_S(s) = \int_{-\infty}^{+\infty} f_D(x) f_T(s-x) dx = \int_0^1 e^{-(s-x)} dx = e^{-s}(e-1),$$

If $0 \leq s \leq 1$: $$f_S(s) = \int_{-\infty}^{+\infty} f_D(x) f_T(s-x) dx = \int_0^s e^{-(s-x)} dx = 1 - e^{-s}.$$

(c) The conditional PDF is given by

$$f_{T|S}(t|s) = \frac{f_{S,T}(s,t)}{f_S(s)} = \begin{cases} \frac{e^{-t}}{1-e^{-s}} & \text{if } 0 \leq s \leq 1,\ 0 \leq t \leq s \\ \frac{e^{-t}}{e^{-s}(e-1)} & \text{if } s > 1,\ s-1 \leq t \leq s \\ 0 & \text{otherwise} \end{cases}$$

4. Here is a way to solve Buffon's needle problem without calculus. Recall that an $\ell$ inch needle is dropped at random onto a lined sheet, where the lines are one inch apart.

   (a) Let $A$ be the number of lines that the needle hits. Let $B$ be the number of times that a polygon of perimeter $\ell$ hits a line. Show that $\mathrm{E}[A] = \mathrm{E}[B]$. (**Hint:** Use linearity of expectation.)

   (b) Assume that $\ell < \pi$. Calculate the expected number of times that a circle of perimeter $\ell$ hits a line.

   (c) Assume that $\ell < 1$. Use part (a) and (b) to derive a formula for the probability that the needle hits a line. (**Hint:** The number of hits is a Bernoulli random variable.)

**Solution:**

(a) Suppose the polygon has $n$ edges of length $a_1, a_2, \ldots, a_n$. Break up the needle into segments of lengths $a_1, a_2, \ldots, a_n$. Let $A_i$ and $B_i$ be the number of lines hit by the $i$-th segment of the needle and the $i$-th edge of the polygon, respectively. Then

$$A = A_1 + \cdots + A_n \quad \text{and} \quad B = B_1 + \cdots + B_n.$$

By linearity of expectation

$$\mathrm{E}[A] = \mathrm{E}[A_1] + \cdots + \mathrm{E}[A_n] \quad \text{and} \quad \mathrm{E}[B] = \mathrm{E}[B_1] + \cdots + \mathrm{E}[B_n].$$

Since the $i$-th edge of the polygon and the $i$-th segment of the needle are identical, $\mathrm{E}[A_i] = \mathrm{E}[B_i]$. It follows that $\mathrm{E}[A] = \mathrm{E}[B]$.

(b) Let $C$ be the number of times a circle intersects a line. We calculate the PMF of $C$. Let $d$ be the line segment representing the diameter of the circle that is perpendicular to the lines on the sheet. Since $\ell < \pi$, the length of $d$ is less than 1. The circle hits a line twice if $d$ crosses a line, once if $d$ touches one of the lines, and zero times if $d$ does not intersect any of the lines. The probability that $d$ crosses a line is exactly the length of $d$, namely $\ell/\pi$, and the probability that $d$ touches a line is zero. Summarizing, the PMF of $C$ is

| $c$ | 0 | 1 | 2 |
|---|---|---|---|
| $\mathrm{P}(C = c)$ | $1 - \ell/\pi$ | 0 | $\ell/\pi$ |

Therefore $\mathrm{E}[C] = 2\ell/\pi$.

(c) If we view the circle as a polygon with infinitely many sides, putting together part (a) and (b) we get that $\mathrm{E}[A] = \mathrm{E}[C] = 2\ell/\pi$. Since $\ell < 1$, the number of times the needle intersects a line is a 0/1 valued random variable, so $\mathrm{E}[A] = \mathrm{P}(A = 1) = \mathrm{P}(\text{the needle hits a line})$. Therefore the probability the needle hits a line is exactly $2\ell/\pi$.