1. A box contains 7 red and 7 blue balls. Two balls are withdrawn randomly. If they are the same color, then you win \$1; if they are different colors, then you lose \$1.

   (a) What is the expected value of the amount you win?

   (b) What is the variance of the amount you win?

   Solution: The sample space S consists of all arrangements of 7 red and 7 blue balls. We assume equally likely outcomes. Let E be the event that the first two balls are of the same colour. The random variable X of interest is the amount you win, namely

$$X = \begin{cases} 1, & \text{if } E \text{ occurs} \\ -1, & \text{if not.} \end{cases}.$$

   (a) Let R be the event that the first ball is red. Then

$$P(E) = P(E \mid R)P(R) + P(E \mid R^c)P(R^c) = 2 \times \frac{6}{13} \times \frac{1}{2} = \frac{6}{13}.$$

   So the random variable X has the following p.m.f.:

x:	-1	1
$f(x)$:	$7/13$	$6/13$

   and

$$E[X] = -1 \times \frac{7}{13} + 1 \times \frac{6}{13} = -\frac{1}{13}.$$

   (b) For $Var[X]$, we have:

$$Var[X] = E[(X - E[X])^2] = \left(\frac{-12}{13}\right)^2 \times \frac{7}{13} + \left(\frac{14}{13}\right)^2 \times \frac{6}{13} = \frac{168}{169}.$$

   Alternatively, we can use the formula $Var[X] = E[X^2] - (E[X])^2$. Since X only takes values -1 and 1, $X^2 = 1$ with probability 1 and so

$$Var[X] = 1 - (E[X])^2 = 1 - (1/13)^2 = 168/169.$$

2. Roll a 4-sided die twice. Let X and Y be the minimum and maximum of the two rolls, respectively. Find the joint PMF of X and Y, their marginal PMFs, and the expected value of $X + Y$.

   Solution: When $x < y$ are different, the event $X = x$ and $Y = y$ can happen in 2 out of 16 possible ways: Either the first toss is x and the second toss is y, or the other way around. When $x = y$ then there is only one possible outcome. All other probabilities are zero. Summarizing, the joint PMF $p_{XY}(x, y)$ is

$x\backslash y$	1	2	3	4
1	1/16	1/8	1/8	1/8
2	0	1/16	1/8	1/8
3	0	0	1/16	1/8
4	0	0	0	1/16

The marginal PMFs are obtained by adding the rows and colums, respectively:

z	1	2	3	4
$p_X(z)$	7/16	5/16	3/16	1/16
$p_Y(z)$	1/16	3/16	5/16	7/16

The expected values are $\mathrm{E}[X] = 30/16$ and $\mathrm{E}[Y] = 50/16$. The expected sum is $\mathrm{E}[X+Y] = \mathrm{E}[X] + \mathrm{E}[Y] = 5$. Another way to see this is that if A and B are the first and second rolls then $X + Y = A + B$, so $\mathrm{E}[X+Y] = \mathrm{E}[A+B] = \mathrm{E}[A] + \mathrm{E}[B] = 2 \cdot 5/2 = 5$.

3. Let p be a number between 0 and 1. Toss a p-biased coin. If the coin comes up heads, toss another fair coin and report the outcome twice (1 for heads, 0 for tails). If the coin comes up tails, report the outcomes of two independent fair coin tosses. Show that the marginal PMFs of your two reports are the same for every p, but the joint PMFs are all different.

   Solution: Let X and Y be the two reports and E be the event that the biased coin comes out heads (which occurs with probability p). We calculate the joint PMF of X and Y using the total probability formula:

$$\mathrm{P}(X = x, Y = y) = \mathrm{P}(X = x, Y = y | E) \cdot p + \mathrm{P}(X = x, Y = y | E^c) \cdot (1 - p).$$

   If E occurs, the event "$X = x$ and $Y = y$" can never occur if $x \neq y$. Otherwise, $X = 1, Y = 1$ and $X = 0, Y = 0$ both occur with probability half. If E does not occur then the events $X = x$ and $Y = y$ are conditionally independent so $\mathrm{P}(X = x, Y = y | E^c) = 1/4$, so

$$\mathrm{P}(X = x, Y = y) = \begin{cases} \frac{1}{2}p + \frac{1}{4}(1-p), & \text{if } x = y \\ \frac{1}{4}(1-p), & \text{if } x \neq y. \end{cases}$$

   The joint PMF of X and Y is:

$x \backslash y$	0	1
0	$\frac{1+p}{4}$	$\frac{1-p}{4}$
1	$\frac{1-p}{4}$	$\frac{1+p}{4}$

   and so the joint PMFs are all different as the value of p changes. On the other hand, the marginal PMFs, which equal the row/column sums, are uniform ($\mathrm{P}(X = 0) = \mathrm{P}(X = 1) = \mathrm{P}(Y = 0) = \mathrm{P}(Y = 1) = 1/2$) and therefore the same for all p.

4. 100 balls are tossed at random into 100 bins. Each ball is equally likely to land in any of the bins, independently of the other balls.

   (a) What is the expected number of bins that receive exactly one ball?

   Solution: Let X_i be an indicator random variable for the event that the i-th bin receives exactly one ball ($X_i = 1$ if this happens, $X_i = 0$ if it doesn't.) Then the number of bins X with exactly one ball is

$$X = X_1 + X_2 + \cdots + X_{100}.$$

   Even though the events $X_1 = 1, \ldots, X_{100} = 1$ are not independent, we can apply linearity of expectation to express $\mathrm{E}[X]$ as

$$\mathrm{E}[X] = \mathrm{E}[X_1] + \cdots + \mathrm{E}[X_{100}].$$

Since X_i is an indicator random variable, $\mathrm{E}[X_i] = \mathrm{P}(X_i = 1)$. The number of balls in bin i is a Binomial$(100, 1/100)$ random variable, so the probability that bin i has precisely one ball is

$$\mathrm{P}(X_i = 1) = 100 \cdot (1/100) \cdot (99/100)^{99}$$

and so $\mathrm{E}[X] = 100 \cdot 100 \cdot (1/100) \cdot (99/100)^{99} = 99^{99}/100^{98} \approx 36.973$.

(b) What is the expected number of balls that are not alone in their bin?

Solution: Let X and Y be the number of balls that are alone and not alone in their bin. Because there are 100 balls in total, we have $X + Y = 100$. The number X of balls that are alone in their bin is exactly the same as the number (conveniently also named X) of bins that receive exactly one ball. By linearity of expectation and by the calculation in part (a),

$$\mathrm{E}[Y] = \mathrm{E}[100 - X] = 100 - \mathrm{E}[X] \approx 100 - 36.973 = 63.027.$$

Alternative solution: Let Y_i be an indicator random variable for the event that the i-th ball is not alone ($Y_i = 1$ if this happens, $Y_i = 0$ if it doesn't.) Then the number of balls Y that are not alone in their bin is

$$Y = Y_1 + Y_2 + \cdots + Y_{100}.$$

Even though the events $Y_1 = 1, \ldots, Y_{100} = 1$ are not independent, we can apply linearity of expectation to express $\mathrm{E}[Y]$ as

$$\mathrm{E}[Y] = \mathrm{E}[Y_1] + \cdots + \mathrm{E}[Y_{100}].$$

Since Y_i is an indicator random variable, $\mathrm{E}[Y_i] = \mathrm{P}(Y_i = 1)$. The event $Y_i = 0$ when ball i is alone in its bin. It happens when all other 99 balls land in a different bin, so by independence $\mathrm{P}(Y_i = 0) = (99/100)^{99}$ and $\mathrm{P}(Y_i = 1) = 1 - (99/100)^{99}$. By linearity of expectation,

$$\mathrm{E}[Y] = 100 \cdot (1 - (99/100)^{99}) \approx 63.027.$$