1. Alice rolls three 3-sided dice. Calculate the PMFs and the expected values of

   (a) The maximum of the three rolls.

   Solution: Call this random variable MAX. The sample space has $3^3 = 27$ equally likely outcomes consisting of the three rolls. Out of these, the event $Max = 1$ happens for a single outcome `111`, while $MAX = 2$ happens for all but one of the eight outcomes $\{\mathtt{1}, \mathtt{2}\} \times \{\mathtt{1}, \mathtt{2}\} \times \{\mathtt{1}, \mathtt{2}\}$. Therefore $\mathrm{P}(MAX = 1) = 1/27$ and $\mathrm{P}(MAX = 2) = 7/27$. Since probabilities must add up to one, $\mathrm{P}(MAX = 3) = 19/27$.

   (b) The minimum of the three rolls.

   Solution: You can calculate this as in part (a) or reason it out like this: If we replaced roll x by $3 - x$, the minimum MIN would become $3 - MAX$. Since the replacement preserves the probabilities of all outcomes, MIN and $3 - MAX$ must have the same PMF, which is

x	1	2	3
$\mathrm{P}(MIN = x)$	19/27	7/27	1/27.

   (c) The average of the three rolls.

   Solution: Let's first calculate the PMF of the sum SUM of the three rolls. This can take values $3, 4, 5, 6, 7, 8$, or 9. The event $SUM = 3$ consists of the single outcome `111`, $SUM = 4$ consists of the three outcomes `211`, `121`, `112`, and $SUM = 5$ consists of six outcomes: Three with one `3` roll and two `1` rolls, and three with two `2` rolls and one `1` roll. So $\mathrm{P}(SUM = 3) = 1/27, \mathrm{P}(SUM = 4) = 3/27$, and $\mathrm{P}(SUM = 5) = 6/27$. By the same argument is in part (b), if we replace roll x by $3-x$, SUM becomes $9-SUM$ and so we can deduce the PMF values at 7, 8, and 9. It remains to determine $\mathrm{P}(SUM = 6)$ which must then equal $1-2(1/27-3/27-6/27) = 7/27$. As the average AVE is $SUM/3$ we get the following PMF:

x	1	4/3	5/3	2	7/3	8/3	3
$\mathrm{P}(AVE = x)$	1/27	3/27	6/27	7/27	6/27	3/27	1/27.

   (d) **(Optional)** The median of the three rolls.

   Solution: For the median MED to be 1, at least two of the three rolls must be ones and the third can be arbitrary. The number of outcomes in which two rolls are `1` and the third is *different* is $3 \cdot 2 = 6$, so $\mathrm{P}(MED = 1) = 7/27$. By the same reasoning $\mathrm{P}(MED = 3) = 7/27$, so $\mathrm{P}(MED = 2) = 13/27$.

2. Suppose the number of school bus arriving at the Sir Run Run Shaw Hall in any time interval is a Poisson random variable, with a rate of 1 bus in 5 minutes.

   (a) What is the probability that no bus arrives in an interval of 30 minutes?

   Solution: The rate of bus arrivals is 6 in 30 minutes, so the number of buses that arrive in a 30-minute interval is a Poisson(6) random variable X. We are interested in the probability of the event $X = 0$, which equals e^{-6}.

(b) What is the probability that there are at least 5 buses in an interval of 10 minutes?

Solution: The rate of arrivals is 2 in 10 minutes, so we want to know what is the probability that a Poisson(2) random variable Y takes value 5 or more. So we need to calculate

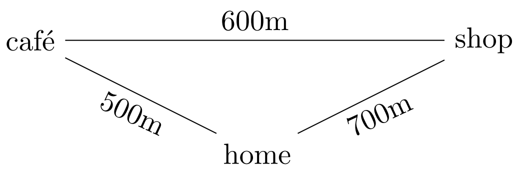
$$\mathrm{P}(Y \geq 5) = \mathrm{P}(Y = 6) + \mathrm{P}(Y = 7) + \cdots ,$$

which is an infinite sum. By the axioms of probability, we can instead calculate

$$\begin{aligned}\mathrm{P}(Y \geq 5) &= 1 - \mathrm{P}(Y < 5)\\ &= 1 - \mathrm{P}(Y = 0) - \mathrm{P}(Y = 1) - \mathrm{P}(Y = 2) - \mathrm{P}(Y = 3) - \mathrm{P}(Y = 4)\\ &= 1 - \frac{e^{-2} \cdot 2^0}{0!} - \frac{e^{-2} \cdot 2^1}{1!} - \frac{e^{-2} \cdot 2^2}{2!} - \frac{e^{-2} \cdot 2^3}{3!} - \frac{e^{-2} \cdot 2^4}{4!},\\ &= 1 - 7e^{-2}\end{aligned}$$

which is about 0.0527.

3. Alice can't find her expensive sweater. She estimates that there is a 30% chance that she left it at the café and a 40% chance that she left it at the shop (and that it is lost with the remaining probability). The distances between her home, the café, and the shop are given below. On her trip to find the sweater, in which order should she visit the venues so as to minimize her expected round-trip walking distance?



Solution: Let C and S be the events that the sweater is at the café and at the shop, respectively, and X be her walking distance which is a random variable. If Alice walks to the café first, her expected walking distance is

$$\mathrm{E}[X] = 2 \cdot 500 \cdot \mathrm{P}(C) + (500 + 600 + 700) \cdot \mathrm{P}(C^c) = 1000 \cdot 0.3 + 1800 \cdot 0.7 = 1560,$$

and if she walks to the shop first it is

$$\mathrm{E}[X] = 2 \cdot 700 \cdot \mathrm{P}(S) + (500 + 600 + 700) \cdot \mathrm{P}(S^c) = 1400 \cdot 0.4 + 1800 \cdot 0.6 = 1640.$$

Therefore Alice should try the café first.

4. Calculate the PMFs of the following random variables:

(a) The first time X at which both the patterns `TH` and `HT` have appeared in a sequence of fair coin flips. For example, $X = 6$ for the sequence `HHTTTH`.

Solution: The events $X = 1$ and $X = 2$ never occur. For $x \geq 3$, the event $X = x$ occurs when either there is a head at time x, and the first $x - 1$ flips consist of some heads followed by some tails with at least one of each kind, or vice versa. As there are $x - 2$ possible positions in which the first flip can occur, there are $x - 2$ possible outcomes of each kind. As each outcome has probability 2^{-x}, $\mathrm{P}(X = x) = 2(x - 2)/2^x$ for $x \geq 3$.

(b) The first time Y at which all three face values have appeared in a sequence of rolls of a fair 3-sided die. For example, $Y = 6$ for the sequence `232231`.

Solution: The events $Y = 1$ and $Y = 2$ never happen. For $y \geq 3$, let R_y be the value of the y-th roll. Conditioned on $R_y = 1$, the event $Y = y$ happens when the first $y - 1$ rolls are all `2`s or `3`s with at least one of each type. Let A be the event "the first $y - 1$ rolls are all `2`s or `3`s" and B be the event "the first y rolls are all of the same type". Then $\mathrm{P}(A) = (2/3)^{y-1}$ by independence of the rolls and $\mathrm{P}(B) = 2/3^{y-1}$ as there are only two outcomes in B. By the axioms of probability,

$$\mathrm{P}(Y = y | R_y = 1) = \mathrm{P}(A \cap B^c) = \mathrm{P}(A) - \mathrm{P}(A \cap B) = (2/3)^{y-1} - 2/3^{y-1}.$$

As $Y = y$ and $R_y = 1$ are independent events, $\mathrm{P}(Y = y) = (2/3)^{y-1} - 2/3^{y-1}$ for $y \geq 3$.