ENGG 2760A / ESTR 2018: Probability for Engineers

Tutorial 9

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Quiz 7:

Alice is about to take two laps in the CUHK swimming pool. The time of her first lap is F minutes, where F is an Exponential(1) random variable. The time of her second lap is S minutes, where S is an Exponential(F) random variable. What is the probability that she completes her second lap within one minute?

$$\underbrace{P(S\le 1)}_{\text{CDF of } S} = \int_0^\infty P(S\le 1, F=f)\,df \quad \text{(total Prob. Theorem)} \quad ①$$

$$= \int_0^1 P(S=s)\,ds \quad \text{(the definition of CDF)} \quad ②$$

① method: $F\sim \text{Exponential}(1) \Rightarrow P(F=f) = \begin{cases} e^{-f} & f\ge 0 \\ 0 & o/w \end{cases}$

$S|F \sim \text{Exponential}(F)$ $\quad P(S=s|F=f) = \begin{cases} fe^{-fs} & s\ge 0 \\ 0 & o/w \end{cases}$

$$\int_0^\infty P(S\le 1, F=f)\,df = \int_0^\infty P(S\le 1|F=f)\cdot P(F=f)\,df$$

$P(S\le 1|F=f)$ $= \int_0^1 fe^{-fs}\,ds$ $= -e^{-fs}\Big|_0^1 = 1-e^{-f}$

CDF of $S|F$

$$= \int_0^\infty (1-e^{-f})e^{-f}\,df = \int_0^\infty (e^{-f}-e^{-2f})\,df$$

$$= -e^{-f} + \frac{1}{2}e^{-2f}\Big|_0^\infty = 0-(-1+\frac{1}{2}) = \frac{1}{2}$$

② $\int_0^1 P(S=s)\,ds = \int_0^1 \int_0^{+\infty} P(S=s|F=f)P(F=f)\,df\,ds$

$$= \int_0^1\int_0^{+\infty} fe^{-fs}\cdot e^{-f}\,df\,ds$$

$$\Rightarrow \quad = \int_0^1\int_0^{+\infty} fe^{-f(s+1)}\,df\,ds$$

$$= \int_0^{+\infty}\int_0^1 f e^{-f\frac{(s+1)}{2}}\,ds\,df = \int_0^{+\infty}\left(-e^{-f(s+1)}\Big|_0^1\right)df$$

$$= \int_0^{+\infty}(-e^{-2f}+e^{f})\,df$$

HW9

Q1

(a) A: the event I am infected

$$Q: P(A|T=t) = \frac{P(A) f_{T|A}(t)}{\boxed{f_T(t)}}$$ (Bayes' Rule)

$P(A) = 1\%$ ①

$$\begin{cases} T|A \sim \text{Normal}(37.8,\ 1.0) & ② \\ T|A^c \sim \text{Normal}(36.8,\ 0.5) \end{cases}$$

(total Prob. Theorem)

$$f_T(t) = P(A) f_{T|A}(t) + P(A^c) f_{T|A^c}(t)$$

$$= 0.01 \cdot \frac{1}{\sqrt{2\pi}} e^{-(t-37.8)^2/2} + 0.99 \cdot \frac{1}{\sqrt{2\pi}(0.5)} e^{-(t-36.8)^2/2(0.5)^2}$$ ③

From ①, ②, ③. we can obtain $P(A|T=t)$

(b) $P(A|T=t) > P(A^c|T=t)$, Find t. satisfies

$$\Downarrow$$

$$\frac{P(A) f_{T|A}(t)}{\cancel{f_T(t)}} > \frac{P(A^c) f_{T|A^c}(t)}{\cancel{f_T(t)}}$$

$$\Rightarrow 0.01 \frac{1}{\sqrt{2\pi}} e^{-\frac{(t-37.8)^2}{2}} > 0.99 \frac{1}{\sqrt{2\pi}(0.5)} e^{-\frac{(t-36.8)}{2(0.5)^2}}$$

$$\Rightarrow t > 38.4591 \text{ or } t < 34.4742$$

Q2.(a) X: # of heads in first two flips

Q: $f_{P|X}(p|2)$

$$f_{P|X}(p|x) = \frac{f_P(p) f_{X|P}(x|p)}{f_X(x)}$$ (Baye's Rule)

$\begin{cases} P \sim \text{Uniform}(0,1) \Rightarrow f_P(p) = 1 \quad p \in [0,1] \\ X|P \sim \text{Binomial}(2, p) \Rightarrow f_{X|P}(x|p) = \binom{2}{x} p^x (1-p)^{2-x} \end{cases}$

2: total # of flip; p: p of heads

$$\Rightarrow f_{P|X}(p|2) = \frac{f_{X|P}(2|p)}{f_X(2)} = \frac{p^2}{f_X(2)} = 3p^2$$

PDF of P|X, $p \in [0,1]$

$$\int_0^1 f_{P|X}(p|2)\,dp = 1 \Rightarrow \int_0^1 \frac{p^2}{f_X(2)}\,dp = 1 \Rightarrow f_X(2) = \int_0^1 p^2\,dp = \frac{1}{3}$$

$f_X(2)$: not relate to p

(b) $E[P|X=2] = \int_0^1 p\, f_{P|X}(p|2)\,dp = \int_0^1 3p^3\,dp = \frac{3}{4}$

(c) Y: # of heads in next 2 flips

Q: $f_{Y|X}(2|2)$ ⇐

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\int_0^1 f_{X,Y|P}(x,y|p) f_P(p)\,dp}{f_X(x)=\frac{1}{3}}$$

$\int_0^1 f_{X,Y,P}(x,y,p)\,dp$

$$\Rightarrow f_{Y|X}(2|2) = \int_0^1 3\,\underline{p^2}\cdot \underline{p^2}\,dp = \int_0^1 3p^4\,dp = \frac{3}{5}$$

$\rightarrow$ the next 2 flips are heads

the first 2 flips are heads

$$f_{Y|X}(2,2) = \frac{\int_0^1 f_{X,Y|P}(2,2|p)\cdot f_P(p)\,dp}{f_X(2)=1/3} = \int_0^1 3p^4\,dp = \frac{3}{5}$$

$$f_{X,Y|P}(2,2|p) = p^2\cdot p^2 = p^4$$

the p of heads

$$f_T(t) = P(A)f_{T|A}(t) + P(A^c)f_{T|A^c}(t)$$ total Prob. Theorem

$\hookrightarrow$ A: discrete variable

$$= \int_A f_A(a)f_{T|A}(t)\,da$$

$\hookrightarrow$ A: continuous variable

Q3: X_1: the time for 1st raindrop

X_2: ~ between 1st ~ 2nd ~

X_3: ~ 2nd ~ 3rd ~

$X_i \sim \text{exponential}(1)$

$$= \begin{cases} e^{-t} & t \geq 0 \\ 0 & o/w \end{cases}$$

$Y = X_1 + X_2$

$\Rightarrow X_2 = Y - X_1$

$$\left.\begin{array}{l} X_1 \geq 0,\ X_2 \geq 0 \\ Y - X_1 \geq 0 \Rightarrow X_1 \leq Y \end{array}\right\} \Rightarrow X_1 \in [0, Y]$$

Convolution formula

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(x_1) f_{X_2}(y-x_1)\,dx_1$$

$$= \int_0^y e^{-x_1}\cdot e^{-(y-x_1)}\,dx_1 = \int_0^y e^{-y}\,dx_1 = ye^{-y} \quad (y \geq 0)$$

$Z = Y + X_3 \Rightarrow X_3 = Z - Y \geq 0 \Rightarrow y \leq z$

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(y) f_{X_3}(z-y)\,dy = \int_0^z y e^{-y} \cdot e^{-(z-y)}\,dy$$

$$= \int_0^z y e^{-z}\,dy = \frac{z^2}{2} e^{-z} \quad (z \geq 0)$$

Q4. $X, Y \sim \text{Uniform}(0,1)$, X, Y independent.

(a) $X' = \ln X$ continuous variable CDF $\rightarrow$ PDF

the range of variable

① $P(X' \leq x') = P(\ln X \leq x') = P(X \leq e^{x'})$

CDF

$$f_X(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & o/w \end{cases} \Rightarrow F_X(x) = \begin{cases} 1 & x > 1 \\ x & x \in [0,1] \\ 0 & x < 0 \end{cases}$$

$F_X(x) = \int_0^x 1\,dx$ $x \in [0,1]$ $= x$

if $x > 1$ $F_X(x) = \int_0^1 1\,dx = 1$

if $x < 0$ $F_X(x) = 0$

$$\Rightarrow P(X \leq e^{x'}) = \begin{cases} 1 & e^{x'} > 1 \Rightarrow x' > 0 \\ e^{x'} & e^{x'} \in [0,1] \Rightarrow x' \leq 0 \\ 0 & e^{x'} < 0 \end{cases}$$

PDF

$$f_{X'}(x') = \frac{dP(X' \leq x')}{dx'} = \begin{cases} e^{x'} & x' \leq 0 \\ 0 & o/w \end{cases}$$

(b) $Z = \ln X + \ln Y$, $X' = \ln X$, $Y' = \ln Y$ $\Bigg\} \Rightarrow Z = X' + Y' \Rightarrow Y' = Z - X' \leq 0$

$0 \geq X' \geq Z$

$$f_{X'}(x') = \begin{cases} e^{x'}, & x' \leq 0 \\ 0, & o/w \end{cases} \qquad f_{Y'}(y') = \begin{cases} e^{y'}, & y' \leq 0 \\ 0, & o/w \end{cases}$$

Convolution formula:

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{X'}(x') f_{Y'}(z-x')\,dx'$$

$$= \int_z^0 e^{x'} \cdot e^{(z-x')}\,dx' = \int_z^0 e^z\,dx' = -ze^z$$

$(z \le 0)$

(c) $e^z = XY$: $T = e^z$ Q: $f_T(t)$

$\hookrightarrow z = \ln X + \ln Y$

$z \le 0$

$T = e^z \in [0,1]$

$\Downarrow$

CDF:

$$P(T \le t) = P(e^z \le t) = P(z \le \ln t)$$

$$= \int_{-\infty}^{\ln t} -ze^z\,dz = e^z - ze^z \Big|_{-\infty}^{\ln t}$$

$$= t - t\ln t$$

PDF

$$f_T(t) = \frac{dP(T \le t)}{dt} = 1 - \ln t - 1 = -\ln t \quad t \in (0,1]$$