ENGG 2760A / ESTR 2018: Probability for Engineers

Tutorial 8

1. Quiz 6
2. Homework 8

Quiz 6:

A train's arrival time at Kowloon station is T minutes past noon, where the PDF of T is

$$f_T(t) = \begin{cases} 1/(t+1)^2, & \text{if } t \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Given that the train hasn't arrived by 12:05, what is the probability that it arrives by 12:10?

by = at before

1) Given that ⇒ conditional Prob.

$$P(T \le 10 \mid T > 5) = \frac{P(5 < T \le 10)}{\boxed{P(T > 5)}}$$

2) $\text{CDF} = \int_{-\infty}^{t} f(x)\,dx$

Definite Integral ≠ indefinite integral

let $g(x) = \int f(x)\,dx = -\frac{1}{x+1}$

$$F(t) = \int_{-\infty}^{t} f(x)\,dx = g(x)\Big|_0^t = g(t) - g(0) = -\frac{1}{x+1}\Big|_0^t$$

$g(0) = (-1)$

$$g(t) = -\frac{1}{t+1} - (-1) = 1 - \frac{1}{t+1}$$

3) for continuous variable $P(T=t) = 0$

$f(x)$

PDF vs PMF

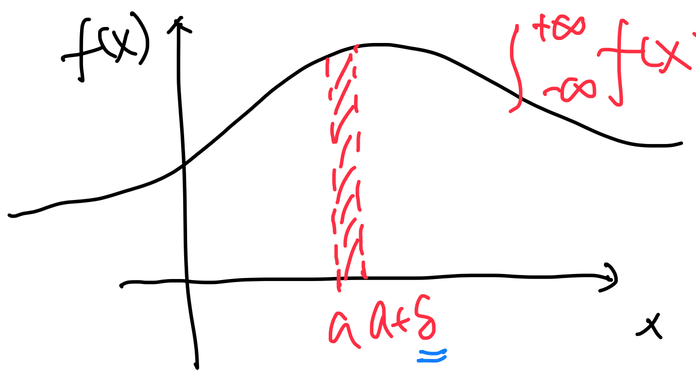
PDF → continuous variables

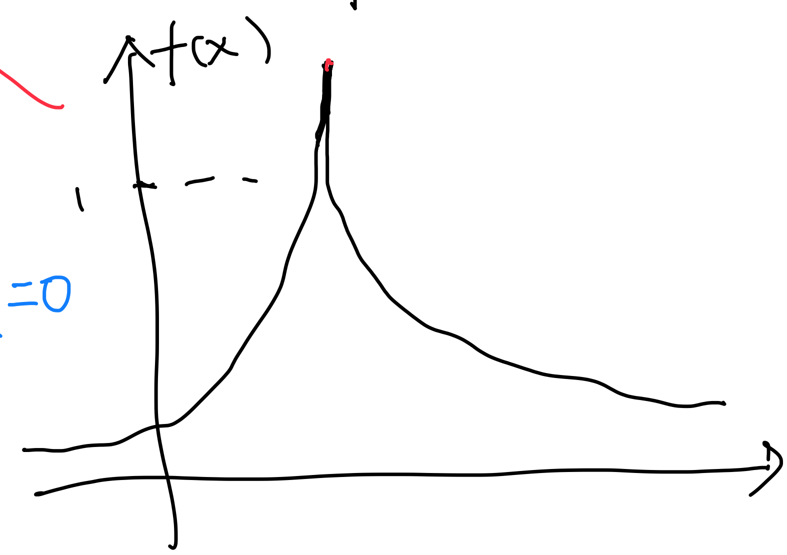
PMF → discrete variable

$P(X=a)$ $\quad 0 \quad f(X=a)$

$\boxed{f(x) \geq 0}$ $\qquad 0 \leq f(x) \leq 1$ for any $x \in X$

$\int_{-\infty}^{+\infty} f(x)\,dx = 1 \Longleftrightarrow \sum_{x \in X} f(x) = 1$





$P(X=a) = \lim_{\delta \to 0} F(a+\delta) - F(a) \approx f(a) \cdot \delta = 0$

HW8

1. (a) $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = 1 \Rightarrow C = \frac{3}{40}$

(b) $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$ ⇐ conditional prob. theorem.

$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \int_0^2 C(x+y+1)y\,dy = C(2x + \frac{14}{3})$

$f_{Y|X}(y|x) = \frac{(x+y+1)y}{2x + \frac{14}{3}}$

2.(a) $\frac{7}{9}$

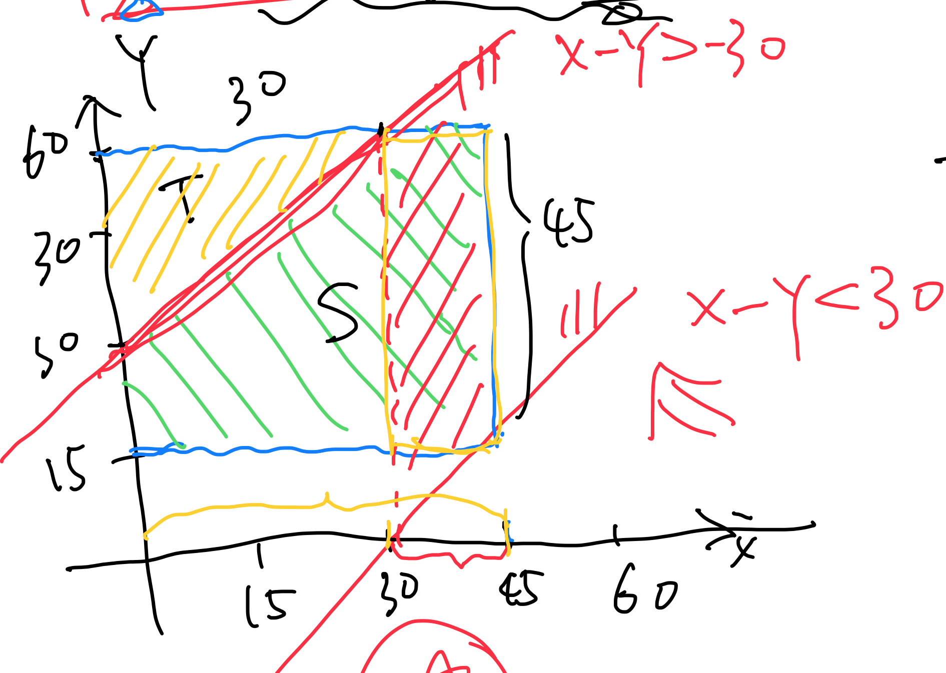
X : Alice arrival minitues after 12:00, $X \sim \text{Uniform}(0,45)$

Y : Bob ~ $\quad Y \sim \text{Uniform}(15,60)$

① $\boxed{f_{X,Y}(x,y)} = f_X(x)\cdot f_Y(y) = \begin{cases} \frac{1}{45^2} & x\in[0,45),\ y\in[15,60) \\ 0 \end{cases}$

②

$\boxed{E : |X-Y| < 30} \Rightarrow -30 < X-Y < 30$



$P(E) = P(S) = 1 - P(T)$

$= 1 - \frac{\text{Area}(T)}{\text{Area}(T+S)} = 1 - \frac{30^2/2}{45^2}$

$= 1 - \frac{2}{9} = \frac{7}{9}$

(b)/(c) Z : $X \in [30,45)$ $\quad X \sim \text{Uniform}(0,45)$

$P(E) = P(E|Z)P(Z) + \boxed{P(E|Z^c)}P(Z^c)$

$= 1 \times \frac{45-30}{45} + P(-30 < X-Y < 30 \mid X < 30) \times \frac{2}{3}$

Q

$P(\sim) \in [0, 1]$ $\quad P(E) \in [\frac{1}{3}, 1]$

$P(\sim) = 0 \quad P(\sim) = 1$

Let $\underbrace{Y=X+15}$ $\quad X\sim \text{Uniform}(0,45)$

$Y\sim \text{Uniform}(15,60)$

$\underbrace{P(Q)=1 \Rightarrow P(E)=1}$

Let $Y=X+30$ $\underbrace{\text{for } X\in(0,30)}$ $\Leftarrow$

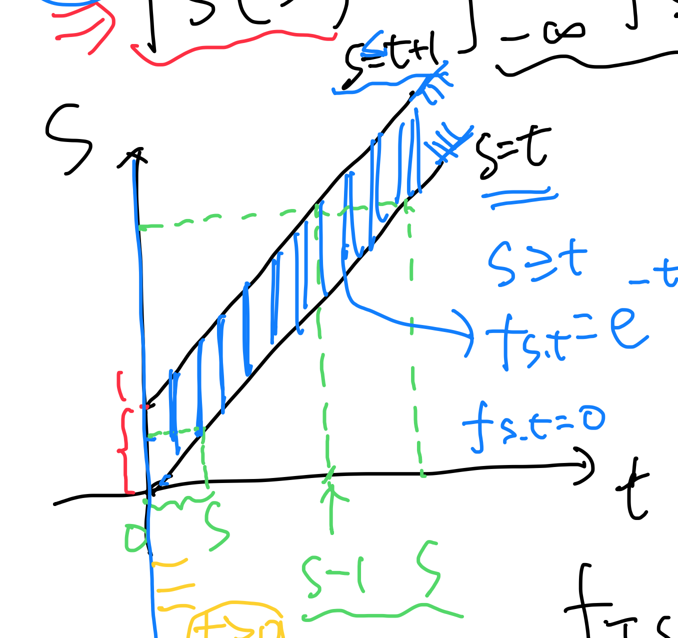
$P(Q)=0 \Rightarrow P(E)=\frac{1}{3}$

3. (a) $\underbrace{T}\sim \text{Exponential}(1) \Rightarrow \underbrace{f_T(t)}=\begin{cases} e^{-t} & t\geq 0\\ 0 & o/w\end{cases}$

$S\sim \underbrace{\text{Uniform}(T,T+1)} \Rightarrow \underbrace{f_{S|T}(s|t)}=\begin{cases}1 & t\leq s\leq t+1\\ 0 & o/w\end{cases}$

$\Rightarrow \boxed{f_{S,T}(s,t)} = f_T(t)f_{S|T}(s|t)=\begin{cases} e^{-t} & \boxed{t\geq 0,\ t\leq s\leq t+1}\\ 0 & o/w\end{cases}$

(b) $\Rightarrow \underbrace{f_S(s)}=\underbrace{\int_{-\infty}^{+\infty} f_{S,T}(s,t)dt} = \begin{cases}\int_0^s e^{-t}dt & s\in[0,1)\\ \int_{s-1}^s e^{-t}dt & s\geq 1\end{cases}$



$s\in[t,t+1]$

$s\in[0,1)$

$= \begin{cases}1-e^{-s} & \\ e^{-s}(e-1) & s\geq 1\end{cases}$

(c) $f_{T|S}(t|s)=\frac{\underbrace{f_{T,S}(t,s)}}{\underbrace{f_S(s)}}=\begin{cases}\dfrac{e^{-t}}{1-e^{-s}} & \text{if } s\in[0,1],\ t\in[0,s]\\ \dfrac{e^{-t}}{e^{-s}(e-1)} & \text{if } s\in(1,+\infty),\ t\in[s-1,s]\\ 0 & o/w\end{cases}$

4.(a) needle: ℓ Polygon: ℓ

Break up into n segments of length $a_1, a_2, \cdots, a_n$ $\Leftarrow$ Assume: n edges, length $a_1, \cdots, a_n$; $a_1+a_2+\cdots+a_n=\ell$

Indicator variable

$A_i = \begin{cases} 1 : & i\text{-th segment hits a line} \\ 0 : & o/w \end{cases}$, $B_i = \begin{cases} 1 : & i\text{-th edge hits a line} \\ 0 & o/w \end{cases}$

Linearity of Expectation

$E[A] = E[A_1] + \cdots + E[A_n]$ $E[B] = E[B_1] + \cdots + E[B_n]$

length of i-th segment = length of i-th edge.

$P(A_i) = P(B_i)$ (for $i=1,\cdots,n$)

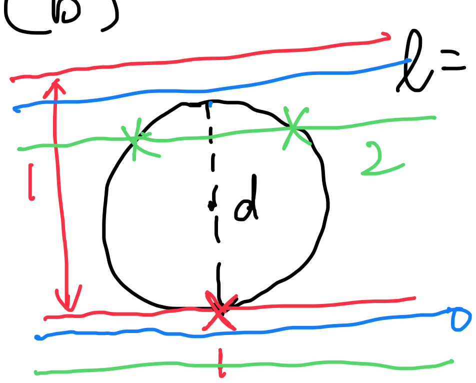
$\Downarrow$

$E[A_i] = E[B_i]$ ~

$\Downarrow$

$E[A] = E[B]$

(b)



$\ell = \pi \cdot d \Rightarrow d = \frac{\ell}{\pi} < 1$

Let C: the number of times intersects a line

C	0	1	2
P	$1-d=1-\frac{\ell}{\pi}$	0	$d=\frac{\ell}{\pi}$

$E[C] = \frac{2\ell}{\pi}$

(c) circle = polygon with infinite edges. $\Rightarrow E[C]=E[B]=\frac{2\ell}{\pi}=E[A]$

$\boxed{\ell<1}$: $A = \begin{cases} 1 & \text{hits a line} \\ 0 & o/w \end{cases}$ $E[A] = P(A) = \frac{2\ell}{\pi}$