CSCI 3130: Formal Languages and Automata Theory — Homework 6
The Chinese University of Hong Kong, Fall 2018 — due 1:59pm Monday December 10

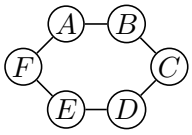
Collaborating on homework is encouraged, but you must write your own solutions in your own words and list your collaborators. Copying someone else's solution will be considered plagiarism and may result in failing the whole course.

Please answer clearly and concisely. Explain your answers. Unexplained answers will get lower scores or even no credits.

(1) (30 points) For each of the following problems, show that it is NP-complete (namely, (1) it is in NP and (2) some NP-complete language reduces to it.) When showing NP-completeness, you can start from any language that was shown NP-complete in class or tutorial.

(a) $L_1 = \{\langle G' \rangle \mid \text{Graph } G' \text{ on } n' \text{ vertices contains an independent set of size } n'/2\}$
Hint: Reduce from Independent Set. When reducing from an instance $\langle G, k \rangle$ of Independent Set, consider the following three cases separately (here n denotes the number of vertices of G): (1) $k = n/2$, (2) $k < n/2$, (3) $k > n/2$.

(b) $L_2 = \{\langle G, k \rangle \mid \text{Graph } G \text{ contains an induced matching of } k \text{ edges}\}$.
An edge subset $M = \{e_1, \ldots, e_k\}$ in G is an induced matching if $u \in e_i$ and $v \in e_j$ for $i \neq j$ implies u and v are non-adjacent (i.e. vertices from different edges in M are non-adjacent). In the graph on the right, $\{(A, B), (D, E)\}$ is an induced matching but $\{(A, B), (C, D)\}$ is not.



Hint: Reduce from Independent Set. Attach new edges to vertices. Be specific about how to do so.

(2) (20 points) For each of the following languages, suppose some polynomial-time algorithm A decides the corresponding *decision* problem. Using A, give a polynomial-time algorithm to *search* for a solution, whenever such a solution exists.

Explain why your algorithm works. Insufficient explanation will get zero points.

(a) $\text{VC} = \{\langle G, k \rangle \mid \text{Graph } G \text{ contains a vertex cover of size } k\}$

(b) $\text{IM} = \{\langle G, k \rangle \mid \text{Graph } G \text{ contains an induced matching of } k \text{ edges}\}$
Induced matching is defined in Q1(b).

(3) (25 points) Throughout the semester, we looked at various models of computation and we came up with the following "hierarchy" of classes of languages:

$$\text{regular} \subseteq \text{context-free} \subseteq \text{P} \subseteq \text{NP} \qquad \text{decidable} \subseteq \text{recognizable}$$

We also gave examples showing that the containments are strict (e.g., a context-free language that is not regular), except for the containment $\text{P} \subseteq \text{NP}$, which is not known to be strict.

There is one gap in this picture between NP languages and decidable languages. In this problem you will fill this gap.

(a) Show that 3SAT is decidable, and the decider has running time $2^{O(n)}$. (Unlike a *verifier* for 3SAT which is given a 3CNF formula φ together with a potential satisfying assignment for φ, a *decider* for 3SAT is only given a 3CNF formula but not an assignment for it.)

(b) Argue that for every NP-language L there is a constant c such that L is decidable in time $2^{O(n^c)}$. (Use the Cook–Levin Theorem.)

(c) Let L' be the following language:

$$L' = \{\langle M, w\rangle \mid \text{Turing machine } M \text{ does not accept input } \langle M, w\rangle \text{ within } 2^{2^{|w|}} \text{ steps}\}.$$

It is not hard to see that L' can be decided in time $O(2^{2^n})$.

Show that L' cannot be decided in time $2^{O(n^c)}$ for any constant c, and therefore it is not in NP.

Hint: Assume that L' can be decided by a Turing machine D in time $2^{O(n^c)}$. What happens when D is given input $\langle D, w\rangle$, where w is a sufficiently long string?

(4) (25 points) A *heuristic* is an algorithm that often works well in practice, but it may not always produce the correct answer. In this problem, we will consider a heuristic for VC (vertex cover).

Recall that the *degree* of a vertex is the number of edges incident to it. In the following, we assume the vertices in the input graph G are labelled from 1 to n. Consider the following heuristic A for VC:

Algorithm 1 GreedyVertexCover(G, k)

Require: G is a graph, k is a nonnegative integer

1: Set $S = \emptyset$ and $H = G$
2: **while** H has at least one edge **do**
3: Set v to be a maximum degree vertex in H
▷ if there are more than one choice for v
▷ break ties by choosing the vertex with the smallest label
4: Add v to S
5: Remove v and all edges incident to v from H
6: **end while**
7: **accept** if and only if $|S| \leqslant k$

(a) Show that A runs in polynomial time.

(b) Show, using a loop invariant, that S is guaranteed to be a vertex cover of G at the end of the while loop.
(Note: As a result, if A accepts $\langle G, k\rangle$, then $\langle G, k\rangle \in$ VC)

(c) Show that it is possible that A rejects $\langle G, k\rangle$, even though $\langle G, k\rangle \in$ VC.
Give such an instance $\langle G, k\rangle$ where the graph G contains at most 5 vertices.