Problem 1

Show $E_{TM} = \{M : M \text{ is a TM and } L(M) = \emptyset\}$ is undecidable.

Solution

We have already known that $L = \{(w, M) : w \text{ is accepted by M.}\}$ is undecidable. Suppose E_{TM} is decidable, then there exists a TM, say A, that can decide E_{TM}. For any input (w, M), construct M' as follows.

(i) Rejects if the input does not equal to w.

(ii) If the input equals to w, simulate M on w. Accepts if M rejects w and rejects if M accepts w.

Now, we show that A accepts M' iff M accepts w. "$\Rightarrow$": If A accepts M', then M' rejects all strings including w and this implies M accepts w. "$\Leftarrow$": If M accepts w, then M' rejects w. Since M' also rejects all the other strings, M' is accepted by A.

Now, we construct a TM for L. On input (w, M), construct M' and run A on it, accepts if and only if A accepts. This contradict to the fact that L is undecidable.

Problem 2

$EQ = \{(M_1, M_2) : M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$ is undecidable.

Solution

Suppose EQ is decidable, then there exists a TM, say A, that can decide EQ. Now, we use A to construct a TM to decide L, which is a contradiction. M_1 rejects all strings. For input (w, M), construct M_2 as follows,

(i) Rejects if the input does not equal to w.

(ii) If the input equals to w, simulate M on w. Accepts if M rejects w and rejects if M accepts w.

Since the definition of M_2 is the same as that of M' in problem 1, then $L(M_2) = \emptyset$ is equivalent to M accepts w, and M_2 rejects all strings is trivially equivalent to $L(M_2) = L(M_1)$.

Now, we construct a TM for L. On input (w, M), construct (M_1, M_2) and run A on it, accepts if and only if A accepts. This contradict to the fact that L is undecidable.

Problem 3

Show $T = \{M : \ M \text{ is a TM that accepts } w^R \text{ whenever it accepts } w\}$ is undecidable.

Solution

We have already known that $L = \{(w, M) : \ w \text{ is accepted by M.}\}$ is undecidable. Suppose T is decidable, then there exists a TM, say A, that can decide T. For any input (w, M), construct M' as follows.

If $w = w^R$, simulate M on w. The alphabet set of M is Σ, without loss of generality, say $a, b \notin \Sigma$. Let $\Sigma \cup \{a, b\}$ be the alphabet set of M'. M' rejects all the other strings other than ab, for input ab, simulate M on w,

(i) If M accepts w, M' rejects.

(ii) If M rejects w, M' accepts.

Now, we show that A accepts M' iff M accepts w. "$\Rightarrow$": If A accepts M', singce M' rejects all the other strings including ba, then M' rejects ab and this implies M accepts w. "$\Leftarrow$": If M accepts w, then M' rejects ab. Since M' rejects all the other strings, M' is accepted by A.

Now, we have constructed a TM for L. On input (w, M), construct M' and run A on it, accepts if and only if A accepts. This contradict to the fact that L is undecidable.