These are the examples for Tutorial 3 with solutions. The alphabet is $\Sigma = \{0, 1\}$ in all the examples.

Problem

Which of these languages is regular?

(a) $L_1 = \{0^m 1^n \colon\ m > n \geq 0\}$

(b) $L_2 = \{0^{2n} \colon n \geq 1\}$

(c) $L_3 = \{0^m 1^n 0^{m+n} \colon m \geq 1 \text{ and } n \geq 1\}$

(d) $L_4 = \{x \colon x \text{ does not have three consecutive 0s}\}$

(e) $L_5 = \{x \colon x \text{ has an equal number of 0s and 1s}\}$

(f) $L_6 = \{x \colon x = x^R\}$. Recall that x^R is x written backwards; for example, $(011)^R = 110$

(g) $L_7 = \{0^{n^2} \colon\ n \text{ is an integer and } n \geq 0\}$

(h) $L_8 = \{0^n \colon n \text{ is a prime}\}$

(i) $L_9 = \{x \colon x \text{ has a different number of 0s and 1s}\}$

The solutions are on the next page.

Solution

(a) We show L_1 is not regular using the pumping lemma. Suppose L_1 is regular. Let n be its pumping length. Take $z = 0^n 1^{n-1}$, which is in L_1. Then u and v consist only of zeros. By the pumping lemma, we can write $z = uvw$ where $|uv| \leq n$ and $|v| \geq 1$ so that $uv^i w \in L_1$ for every i. In particular $uw = uv^0 w$ should by in L_1. But uw has at most $n-1$ zeros and at least $n-1$ ones, so $uw \notin L_1$, a contradiction.

(b) L_2 is described by the regular expression $(00)^*$, so it is regular.

(c) We show L_3 is not regular using the pumping lemma. Suppose L_3 is regular. Let n be its pumping length. Take $z = 0^n 1^n 0^{2n}$, which is in L_3. Then u and v consist only of zeros. By the pumping lemma, we can write $z = uvw$ where $|uv| \leq n$ and $|v| \geq 1$ so that $uv^i w \in L_3$ for every i. In particular $uw = uv^0 w$ should by in L_3. But uw has fewer 0s in the first block than 1s in the second block, so it is not in L_3, a contradiction.

(d) The complement of L_4 is the language $\{x\colon x \text{ contains three consecutive 0s}\}$. This language is described by the regular expression $(0+1)^*000(0+1)^*$, so it is regular. Therefore L_4 is also regular.

(e) We show L_5 is not regular using the pumping lemma. Suppose L_5 is regular. Let n be its pumping length. Take $z = 0^n 1^n$, which is in L_5. Then u and v consist only of zeros. By the pumping lemma, we can write $z = uvw$ where $|uv| \leq n$ and $|v| \geq 1$ so that $uv^i w \in L_5$ for every i. In particular $uw = uv^0 w$ should by in L_5. But uw has fewer 0s in the first block than 1s in the second block, so it is not in L_5, a contradiction.

(f) We show L_6 is not regular using the pumping lemma. Suppose L_6 is regular. Let n be its pumping length. Take $z = 0^n 1 0^n$, which is in L_6. Then u and v consist only of zeros. By the pumping lemma, we can write $z = uvw$ where $|uv| \leq n$ and $|v| \geq 1$ so that $uv^i w \in L_6$ for every i. In particular $uw = uv^0 w$ should by in L_6. But uw has the form $0^m 1 0^n$, where $m < n$. So $(uw)^R = 0^n 1 0^m \neq uw$, and uw is not in L_6, a contradiction.

(g) We show L_7 is not regular using the pumping lemma. Suppose L_7 is regular. Let n be its pumping length. Take $z = 0^{n^2}$, which is in L_7. By the pumping lemma, we can write $z = uvw$ where $|uv| \leq n$ and $|v| \geq 1$ so that $uv^i w \in L_7$ for every i. In particular $uv^2 w$ should by in L_7. But $uv^2 w$ has length $n^2 + |v| \leq n^2 + n$, which is a number strictly between n^2 and $(n+1)^2$ (because $(n+1)^2 = n^2 + 2n + 1$), so it is not the square of any number. Therefore $uv^2 w$ is not in L_7, a contradiction.

(h) We show L_8 is not regular using the pumping lemma. Suppose L_8 is regular. Let n be its pumping length. Take $z = 0^p$, where p is any prime bigger than n. (Since there are infinitely many prime numbers, we can always choose such a p.) By the pumping lemma, we can write $z = uvw$ where $|uv| \leq n$ and $|v| \geq 1$ so that $uv^i w \in L_8$ for every i. Take $i = p + 1$. Then uw has length $p - |v|$ and v^i has length $i|v|$. So $uv^i w$ has length $(p - |v|) + i|v| = (p - |v|) + (p-1)|v| = p(|v| + 1)$, which is a product of two numbers greater than one. The length of $uv^i w$ is not a prime number, so $uv^i w \notin L_8$, a contradiction.

(i) The easier way to prove L_9 is not regular goes like this. Suppose it is regular, then L_5 is L_9's complement, hence L_5 is regular. This contradicts part (e).

If you want to prove L_9 is not regular using the pumping lemma, it is also possible, but a bit more difficult. Suppose it is regular and let n be its pumping length. Take $z = 0^n 1^{n+n!}$, which is in L_9. ($n!$ is the factorial of n, given by $n! = 1 \cdot 2 \cdot 3 \ldots n$.) By the pumping lemma, we can write $z = uvw$ where $|uv| \leq n$ and $|v| \geq 1$ so that $uv^iw \in L_9$ for every i. But if we set $i = n!/|v| + 1$ (which is an integer because $|v| \leq n$, and so it divides $n!$), we get that uv^iw has $n + (i-1)|v|$ zeros and $n!$ ones. By our choice of i, $uv^iw = 0^{n!}1^{n!} \notin L_9$, a contradiction.