You are encouraged to work on this homework together as long as you write up your own solutions. If you do so, write the names of your collaborators. Please refrain from looking up solutions to the homework on the internet or in other sources. If you must, state the source.

Problem 1

Consider the following dictatorship test:

> Given a function $f\colon \{0,1\}^n \to \{0,1\}$:
> Apply the linearity test to f:
> Choose random $a, b \sim \{0,1\}^n$ and reject if $f(a) + f(b) \neq f(a+b)$.
> Choose random $x, y \sim \{0,1\}^n$ and a random partition (I, J) of $\{1, \dots, n\}$.
> If $f(x_I x_J) \neq f(x_I y_J)$ and $f(x_I x_J) \neq f(y_I x_J)$, reject.
> Otherwise, accept.

A random partition (I, J) of $\{1, \dots, n\}$ is chosen by including each element in I independently and uniformly at random and setting J to be the complement of I. The notation $z_I w_J$ is used for a string in $\{0,1\}^n$ whose ith coordinate is z_i if $i \in I$ and w_i if $i \in J$.

(a) Show that if f is a dictator, i.e. $f(x) = x_i$ for some $i \in \{1, \dots, n\}$, then the test accepts f.

(b) Show that if f is balanced (i.e. $\mathrm{E}_{x\sim\{0,1\}^n}[f(x)] = 1/2$) and the test accepts f with probability $1 - \delta$, then there exists a dictator x_i such that $\Pr_{x\sim\{0,1\}^n}[f(x) = x_i] = 1 - O(\delta)$.

Problem 2

The *double cover* of a graph G is the graph G_2 that has two copies v_1, v_2 of every vertex v in G and an edge (v_1, w_2) for every edge (v, w) of G. For example the double cover of the 3-cycle is the 6-cycle. Show that $\lambda_2(G_2) = \max(\lambda_2(G), -\lambda_n(G))$. Here λ_2 and λ_n denote the second smallest and smallest eigenvalues of the corresponding graph.

Problem 3

Let $D \subseteq \{0,1\}^n$ be an ε-biased distribution and G be a regular graph whose vertices are labeled by samples of D so that the number of vertices labeled x is proportional to the probability of x under D. Let D' be the following distribution: Uniformly choose a random edge (x_1, x_2) of G and output $x_1 + x_2$. Show that D' is $(\varepsilon^2 + \lambda)$-biased, where $\lambda = \max(\lambda_2(G), -\lambda_n(G))$.