A standard deck of 52 cards is split evenly among 26 people. Let X be the number of people whose two cards have the same face value (e.g., $\{9\spadesuit, 9\heartsuit\}$). Calculate $\mathrm{E}[X]$ and $\mathrm{Var}[X]$.

Solution: Let X_i be the indicator variable that the two cards of the i-th person have the same face value. Then $X = X_1 + X_2 + \cdots + X_{26}$. By linearity of expectation,

$$\mathrm{E}[X] = \mathrm{E}[X_1] + \mathrm{E}[X_2] + \cdots + \mathrm{E}[X_{26}],$$

where $\mathrm{E}[X_i] = \mathrm{P}(X_i = 1)$ is the probability that a person gets two cards of the same face value. There are 3 good choices for the second card out of 51 and so $\mathrm{E}[X_i] = 3/51$ and $\mathrm{E}[X] = 26/17$. For the variance we use the sum-of-covariances formula:

$$\mathrm{Var}[X] = \sum_{i=1}^{26} \mathrm{Var}[X_i] + \sum_{i \neq j} \mathrm{Cov[X_i X_j]}.$$

As usual, $\mathrm{Var}[X_i] = 1/17 - (1/17)^2 \approx 0.0553$. For the covariances,

$$\mathrm{Cov}[X_i X_j] = \mathrm{E}[X_i X_j] - \mathrm{E}[X_i]\,\mathrm{E}[X_j] = \mathrm{P}(X_i = 1 \text{ and } X_j = 1) - \mathrm{P}(X_i = 1)\,\mathrm{P}(X_j = 1).$$

The event $X_i = 1$ and $X_j = 1$ happens when both person i and person j obtained two cards of the same value. Conditioned on this happening for person i, there are two possibilities for person j: Either they get the remaining two cards of the same face value as person i, or they get two cards of different face value. The first event has conditional probability $(2/50)(1/49)$ and the second one has conditional probability $(48/50)(3/49)$. Therefore

$$\mathrm{P}(X_i = 1 \text{ and } X_j = 1) - \mathrm{P}(X_i = 1)\,\mathrm{P}(X_j = 1) = \frac{3}{51} \cdot \left(\frac{2}{50} \cdot \frac{1}{49} + \frac{48}{50} \cdot \frac{3}{49}\right) - \left(\frac{1}{17}\right)^2 \approx 4.5195 \cdot 10^{-5}.$$

We conclude that

$$\mathrm{Var}[X] \approx 26 \cdot 0.0553 + 26 \cdot 25 \cdot 4.5195 \cdot 10^{-5} \approx 1.4688.$$