1. X and Y are independent random variables, both with the following PMF:

x	1	2	4
$f(x)$	1/3	1/3	1/3

(a) Find the PMF of $X + Y$.

Solution: Let $Z = X+Y$. Using the convolution formula $\mathrm{P}(Z = z) = \sum_x \mathrm{P}(X = x)\,\mathrm{P}(Y = z - x)$ we obtain the following PMF:

z	2	3	4	5	6	8
$f_Z(z)$	1/9	2/9	1/9	2/9	2/9	1/9

(b) Are X and $X + Y$ independent? Justify your answer.

Solution: No, because $\mathrm{P}(X = 1, X + Y = 3) = \mathrm{P}(X = 1, Y = 2) = \mathrm{P}(X = 1)\,\mathrm{P}(Y = 2) = 1/9$, while $\mathrm{P}(X = 1)\,\mathrm{P}(X + Y = 3) = 1/3 \cdot 2/3 = 2/9$.

2. The number of cars behind a traffic light at the time it turns green is a Poisson random variable X with mean 1. The number of cars that cross the green light is $\min\{X, 3\}$.

(a) Find the PMF of the number of cars that cross the (green) light.

Solution: Let Y be this number. Then Y and X have the same probability of taking values 0, 1, and 2. Since probabilities must add up to one the event $Y = 3$ must be assigned the remaining probability. Using the Poisson PMF formula $\mathrm{P}(X = x) = 1/(x!e)$ we obtain the following PDF for Y:

y	0	1	2	3
$f(y)$	$1/e$	$1/e$	$1/2e$	$1 - 5/2e$.

(b) The light turns green 50 times within the hour. Is the probability that more than 100 cars cross within the hour larger or smaller than 50%? Justify your answer.

Solution: The expected number of cars that cross a green light is

$$\mathrm{E}[Y] = 1 \cdot \frac{1}{e} + 2 \cdot \frac{2}{e} + 3\left(1 - \frac{5}{2e}\right) = 3 - \frac{11}{2e} \approx 0.977.$$

By linearity of expectation, the expected number of cars that cross within the hour is about $50 \cdot 0.977$. By Markov's inequality, the probability that more than $2 \cdot 50 \cdot 0.977 = 97.7$ cars cross within the hour is less than 50%.

(Alternatively you can apply Markov's inequality to the sum of the X's and use the fact that the Y's are never larger than the X's.)

3. Alice and Bob independently arrive at the bus stop at a uniformly random time between 8 and 9. There are buses at 8.15, 8.30, and 9.

   (a) What is the probability that they catch the same bus?

   Solution: We model A and B as independent Uniform$(0, 1)$ random variable representing the fraction of the hour at which Alice show up. The event E of interest is "$A, B \leq 1/4$ or $1/4 < A, B \leq 1/2$ or $A, B > 1/2$". Since the events are disjoint and A, B are independent,

$$\begin{aligned} \mathrm{P}(E) &= \mathrm{P}(A, B \leq 1/4) + \mathrm{P}(1/4 < A, B \leq 1/2) + \mathrm{P}(A, B > 1/2) \\ &= \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{8}. \end{aligned}$$

   (b) Given that Bob did't run into Alice on the 8.30 bus, what is the probability that Alice caught the 8.15 bus?

   Solution: Let F be the event that Alice did not arrive between 8.15 and 8.30, namely "$A \leq 1/4$ or $A > 1/2$". Then

$$\begin{aligned} \mathrm{P}(A \leq 1/4 | F) &= \frac{\mathrm{P}(A \leq 1/4 \text{ and } F)}{\mathrm{P}(F)} = \frac{\mathrm{P}(A \leq 1/4)}{\mathrm{P}(A \leq 1/4) + \mathrm{P}(A > 1/2)} \\ &= \frac{1/4}{1/4 + 1/2} = \frac{1}{3}. \end{aligned}$$

4. The body weight of a random person is a normal random variable with mean 60kg and standard deviation 10kg. The carrying capacity of an elevator is 600kg. If nine people enter the elevator, what is the probability that the weight limit is exceeded? Assume their weights are independent.

   Solution: The weight of all nine people is a sum of nine independent Normal$(60, 10)$ random variables, which is a normal random variable of mean $9 \cdot 60 = 540$ and standard deviation $\sqrt{9} \cdot 10 = 30$. To exceed the carrying capacity the weight has to exceed its mean by more than two standard deviations. From the table this probability is approximately $1 - 0.9772 = 0.0228$, or 2.28%.

5. Bob found a coin on the street. The null hypothesis is that the coin is fair. Bob conjectures the alternative hypothesis that the coin comes up heads 90% of the time. To test, Bob keeps flipping the coin until a tail comes up and then stops. If t or more flips were performed Bob accepts the alternative hypothesis. If not he rejects it.

   (a) How should Bob choose t if he wants a false positive probability (type I error) of at most 8%?

   Solution: The number N of times the fair coin is flipped until a tail is a Geometric$(1/2)$ random variable. A false positive occurs when N is at least t, namely if the first $t - 1$ tosses were all heads, so $\mathrm{P}(N \geq t) = 1/2^{t-1}$. For a false positive error of at most 8%, $t - 1$ should equal 4, so t should equal 5.

   (b) For the choice of t in part (a), what is the false negative probability (type II error)?

   Solution: This is the probability that a Geometric$(1/10)$ random variable takes value less than $t = 5$, which is $1/10(1 + 9/10 + (9/10)^2 + (9/10)^3) \approx 0.344$.