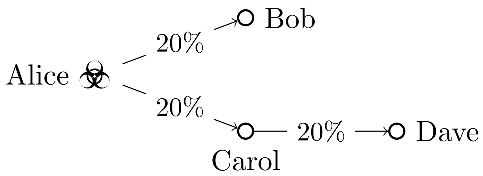
ENGG 2430A: Probability and Statistics for Engineers — Quiz 3 solution
The Chinese University of Hong Kong, Spring 2020

Alice carries a mysterious virus. She meets with Bob and Carol, who later meets with Dave, none of whom initially carry the virus. When carrier x meets non-carrier y, y becomes infected (i.e., y becomes a carrier) with probability 20% independently of all other transmissions. Find the probability mass function of the number of infected people in the group (excluding Alice).



Solution: The number of infected people X is a random variable that can take values in the set $\{0, 1, 2, 3\}$. Let B, C, D be the events that Bob, Carol, and Dave are infected, respectively. Then B and C are independent, and D is conditionally independent of B given C or C^c.

- The event $X = 0$ happens when neither Bob nor Carol are infected. By independence,

$$\mathrm{P}(X = 0) = \mathrm{P}(B^c \cap C^c) = \mathrm{P}(B^c)\,\mathrm{P}(C^c) = 0.8^2 = 0.64.$$

- The event $X = 1$ happens when exactly one among the three is infected. By disjointness and independence,

$$\begin{aligned}\mathrm{P}(X = 1) &= \mathrm{P}\big((B \cap C^c \cap D^c) \cup (B^c \cap C \cap D^c) \cup (B^c \cap C^c \cap D)\big)\\ &= \mathrm{P}(B)\,\mathrm{P}(C^c)\,\mathrm{P}(D^c|C^c) + \mathrm{P}(B^c)\,\mathrm{P}(C)\,\mathrm{P}(D^c|C) + \mathrm{P}(B^c)\,\mathrm{P}(C^c)\,\mathrm{P}(D^c|C^c)\\ &= 0.2 \cdot 0.8 \cdot 1 + 0.8 \cdot 0.2 \cdot 0.8 + 0.8 \cdot 0.8 \cdot 0\\ &= 0.288.\end{aligned}$$

- The event $X = 3$ happens when all of Bob, Carol, and Dave are infected. By conditional indpenendence,

$$\mathrm{P}(X = 3) = \mathrm{P}(B \cap C \cap D) = \mathrm{P}(B)\,\mathrm{P}(C)\,\mathrm{P}(D|C) = 0.2^3 = 0.008.$$

- By the axioms of probability

$$\mathrm{P}(X = 2) = 1 - \mathrm{P}(X = 0) - \mathrm{P}(X = 1) - \mathrm{P}(X = 3) = 0.064.$$

(Alternatively, $\mathrm{P}(X = 2) = \mathrm{P}((B \cap C \cap D^c) \cup \mathrm{P}(B^c \cap C \cap D)) = \mathrm{P}(B)\,\mathrm{P}(C)\,\mathrm{P}(D^c|C) + \mathrm{P}(B^c)\,\mathrm{P}(C)\,\mathrm{P}(D|C) = 0.2 \cdot 0.2 \cdot 0.8 + 0.8 \cdot 0.2 \cdot 0.2 = 0.064$).

In summary, the probability mass function $p(x)$ of X is:

x	0	1	2	3
$p(x)$	0.64	0.288	0.064	0.008