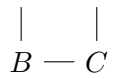


1. Alice, Bob, Charlie, and Dave are randomly seated at a round table. The probability that Alice is seated next to Bob is 70%. The probability that Bob is seated next to Charlie is 40%. What is the probability that Charlie is seated next to Alice?

**Solution:** Let  $A$  be the event “Alice sits across from Dave”. Then the complement of  $A$  is the event “Bob sits next to Charlie”, so  $P(A) = 1 - P(A^c) = 1 - 0.4 = 0.6$ . Define  $B$  and  $C$  analogously with Bob and Charlie replacing Alice, respectively. By the same reasoning  $P(C) = 1 - 0.7 = 0.3$ . Since  $A$ ,  $B$ , and  $C$  partition the sample space,  $P(B) = 1 - P(A) - P(C) = 0.1$ . Therefore the event  $B^c$  representing “Charlie sits next to Alice” has probability  $1 - P(B) = 0.9$ .

2. Computers  $A$ ,  $B$ ,  $C$ , and  $D$  are linked through four cables as in the picture. Each cable  $A - D$  fails with probability 10% independently of the others. Let  $E_{xy}$  be the event “There is a working connection between computers  $x$  and  $y$ .”



- (a) Are  $E_{AB}$  and  $E_{CD}$  conditionally independent given  $E_{BC}$ ?

**Solution:** No. We will show that  $E_{AB}^c$  and  $E_{CD}^c$  are not conditionally independent given  $E_{BC}$ . Since complementing preserves independence  $E_{AB}$  and  $E_{CD}$  cannot be conditionally independent given  $E_{BC}$  either. Let  $F_{xy}$  be the event “Cable  $xy$  fails”. The event  $E_{BC}$  happens when cable  $BC$  works or all of cables  $BA$ ,  $AD$ ,  $DC$  work, so

$$\begin{aligned} P(E_{BC}) &= P(F_{BC}^c \cup (F_{BA}^c \cap F_{AD}^c \cap F_{DC}^c)) = 1 - P(F_{BC}) P(F_{BA} \cup F_{AD} \cup F_{DC}) \\ &= 1 - P(F_{BC})(1 - P(F_{BA}^c) P(F_{AD}^c) P(F_{DC}^c)) = 1 - 0.1 \cdot (1 - 0.9^3) = 0.9729. \end{aligned}$$

The event  $E_{AB}^c \cap E_{CD}^c \cap E_{BC}$  happens when cables  $AB$  and  $CD$  fail but cable  $BC$  doesn't, so it has probability  $0.1^2 \cdot 0.9 = 0.009$ . Therefore

$$P(E_{AB}^c \cap E_{CD}^c | E_{BC}) = \frac{P(E_{AB}^c \cap E_{CD}^c \cap E_{BC})}{P(E_{BC})} = \frac{0.009}{0.9729} \approx 0.0093.$$

The event  $E_{AB}^c \cap E_{BC}$  happens when cable  $BC$  doesn't fail, cable  $AB$  fails, and at least one of cables  $AD$  and  $DC$  fails, so

$$\begin{aligned} P(E_{AB}^c \cap E_{BC}) &= P(F_{BC}^c \cap F_{AB} \cap (F_{AD} \cup F_{DC})) \\ &= P(F_{BC}^c) P(F_{AB})(1 - P(F_{AD}^c) P(F_{DC}^c)) = 0.9 \cdot 0.1 \cdot (1 - 0.9^2) = 0.0171. \end{aligned}$$

Therefore

$$P(E_{AB}^c | E_{BC}) = \frac{P(E_{AB}^c \cap E_{BC})}{P(E_{BC})} = \frac{0.0171}{0.9729} \approx 0.0176.$$

By symmetry,  $P(E_{CD}^c | E_{BC}) = 0.0176$  and so  $P(E_{AB}^c | E_{BC}) P(E_{CD}^c | E_{BC}) \approx 0.0176^2 \approx 0.0003$ , which is much smaller than 0.0093.

- 2 (b) Are  $E_{AB}$  and  $E_{CD}$  conditionally independent given  $E_{BC}^c$  (the complement of  $E_{BC}$ )?

**Solution:** No. As in part (a) we will show that  $E_{AB}^c$  and  $E_{CD}^c$  are not conditionally independent given  $E_{BC}^c$ . From part (a)  $P(E_{BC}^c) = 1 - 0.9729 = 0.0271$ . The event  $E_{AB}^c \cap E_{CD}^c \cap E_{BC}^c$  holds when all 3 cables  $AB, BC, CD$  fail, so  $P(E_{AB}^c \cap E_{CD}^c \cap E_{BC}^c) = 0.1^3 = 0.001$ . Therefore

$$P(E_{AB}^c \cap E_{CD}^c | E_{BC}^c) = \frac{P(E_{AB}^c \cap E_{CD}^c \cap E_{BC}^c)}{P(E_{BC}^c)} = \frac{0.001}{0.0271} \approx 0.0369.$$

On the other hand,  $E_{AB}^c \cap E_{BC}^c$  happens when both cables  $AB$  and  $BC$  fail so  $P(E_{AB}^c \cap E_{BC}^c) = 0.1^2 = 0.01$ . Therefore

$$P(E_{AB}^c | E_{BC}^c) = \frac{P(E_{AB}^c \cap E_{BC}^c)}{P(E_{BC}^c)} = \frac{0.01}{0.0271} \approx 0.369.$$

By symmetry,  $P(E_{CD}^c | E_{BC}^c) \approx 0.0369$  and so  $P(E_{AB}^c | E_{BC}^c) P(E_{CD}^c | E_{BC}^c) \approx 0.1361$ , which is much larger than 0.0369.

3. Alice has a \$1, a \$2, and a \$5 coin. She randomly and secretly picks a coin with each hand (with equal probabilities) and shows the coin in her left hand to Bob. Bob may keep this coin or switch to the coin in Alice's right hand. Assuming Bob plays optimally what is his expected utility?

**Solution:** Let  $L$  and be the value of the coin in Alice's left and hands, respectively. The expectation of  $R$  given  $L$  is the average value of the other two coins:

$$\begin{aligned} E[R|L = 1] &= \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 5 = 3.5 \\ E[R|L = 2] &= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 5 = 3 \\ E[R|L = 5] &= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 2 = 1.5 \end{aligned}$$

To maximize his expected utility, Bob should keep the coin when  $L = 5$  and switch when  $L = 1$  or  $L = 2$ . Since all values of  $L$  are equally likely, Bob's expected utility is

$$\frac{1}{3} \cdot 3.5 + \frac{1}{3} \cdot 3 + \frac{1}{3} \cdot 5 = \frac{23}{6} \approx 3.833.$$

4. Trains reach Kowloon Station at an average rate of two trains per hour. Bob observed at least one train reach the station within the last hour. Given this information, what is the expected number of trains that reached the station within the last hour?

**Solution:** If we model the train arrivals in the last hour as a Poisson(2) random variable  $N$ , we are looking for the expectation of  $N$  given  $N > 0$ . By the total expectation formula

$$E[N] = E[N|N > 0] P(N > 0) + E[N|N = 0] P(N = 0).$$

We know that  $E[N] = 2$ ,  $E[N|N = 0] = 0$ , and  $P(N = 0) = e^{-2} \cdot 2^0/0! = e^{-2}$ . By the axioms  $P(N > 0) = 1 - P(N = 0) = 1 - e^{-2}$ , so

$$E[N|N > 0] = \frac{E[N]}{P(N > 0)} = \frac{2}{1 - e^{-2}} \approx 2.313.$$

5. A dealer divides ten cards with face values  $1, 2, \dots, 10$  among five players. Each player is randomly assigned two cards. A player collects \$1 from the dealer if the sum of his cards' face values is 14 or higher. What is the dealer's expected payout?

**Solution:** The expected payout to player  $i$  is the probability that his cards' face values add up to 14 or higher. A player is equally likely to get any two of the ten cards. By the equally likely outcomes formula, his payout is the number of pairs of cards of value 14 or higher divided by the total number of pairs. The total number of pairs is  $\binom{10}{2} = 45$ , while the pairs of value 14 or higher are  $\{4, 10\}, \{5, 9\}, \{5, 10\}, \{6, 8\}, \{6, 9\}, \{6, 10\}, \{7, 8\}, \{7, 9\}, \{7, 10\}, \{8, 9\}, \{8, 10\}, \{9, 10\}$ . As there are 12 such pairs, the expected payout to player  $i$  is  $12/45$  dollars. By linearity of expectation, the total expected payout is  $5 \cdot 12/45 = 60/45 = 1\frac{1}{3}$  dollars.