On Modeling Product Advertisement in Large Scale Online Social Networks

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***Abstract*—We consider the following advertisement problem in online social networks (OSNs). Given a fixed advertisement investment, e.g., a number of free samples that can be given away to a small number of users, a company needs to determine the probability that users in the OSN will eventually purchase the product. In this paper, we model OSNs as scale-free graphs (either with or without high clustering coefficient). We employ various influence mechanisms that govern the influence spreading in such large scale OSNs and use the local mean field (LMF) technique to analyze these online social networks wherein states of nodes can be changed by various influence mechanisms. We extend our model for advertising with multiple rating levels. Extensive simulations are carried out to validate our models which can provide insight on designing efficient advertising strategies in online social networks.**

***Index Terms*—Local mean field (LMF), online social networks, product advertisement, viral market**

I. INTRODUCTION

In recent years, advertising has become a major commercial activity in the Internet. Traditionally, advertisements are broadcast oriented, e.g., via TV or radio stations so as to reach as many people as possible. With the development of the Internet, new advertisement models emerge and blossom. For example, Google provides the *targeted* advertisements: when a user searches for information, related advertisements, either products or services, are returned together with the search results. Such targeted advertisement can enhance the success rate for selling products. In recent years, online social networks (OSNs) offer another new way of performing advertisement. In OSNs, users are logically grouped together by one or more specific types of interdependency such as friendship, values, interests, ideas, ..., etc. Since the dependency is quite strong, if one user decides to purchase a product, he/she may influence his/her friends, and thereby increases the possibility of sales. With the success of OSNs such as Facebook and Myspace, advertising on OSNs is receiving more attention.

To advertise on OSNs, a company first applies advertising strategies, either traditional or Internet-based, targeted or non-targeted, so as to attract a small fraction of users to purchase the product. Based on this initial fraction of buyers, a cascade of word-of-mouth influence by users is triggered, and eventually large fraction of users may decide to purchase the product. *The aim of our paper is to model advertisement in OSNs. In particular, given a small fraction of users who have purchased the product, what is the influence spread of such a cascade in OSNs and at the steady state, what is the fraction of users in this OSN that will eventually purchase the product?* Predicting the final fraction of buyers is important for companies since one can use this result to design efficient advertising strategies so as to maximize their revenue. However, this is not an easy task since various factors make the analysis difficult. The first important factor is the topology of OSNs which are very different from traditional random graphs. The second important factor is that the mechanism that determines whether a user will purchase a product is unknown. Several conventional models such as the independent cascade model and the linear threshold model [1] characterize such mechanisms, and we will employ them in our analysis later. Thirdly, realistic OSNs are usually large in size (e.g., with over ten million nodes), which makes the analysis complicated.

The contributions of this paper are:

- We use *local mean field* (LMF) to estimate the influence in large social networks. Using the LMF, one can concentrate on the correlation structure of local neighborhoods only, so that one can easily derive the statistical properties of the underlying graphs.
- We formally analyze various influence mechanisms and propose a framework to find the final fraction of buyers under a given mechanism for large social networks. We also validate our models via extensive simulations.
- We extend the analysis to scale-free graphs with high clustering coefficient and propose a framework to quantify the influence in such networks.
- We extend our framework to allow users to have multiple levels of ratings on a product and also show its effectiveness via simulation.

The outline of this paper is as follows. In Section II, we model the underlying OSN as an infinite scale-free random graph and introduce the LMF model, then we present several influence models and estimate the final fraction of buyers. We also validate our analysis via simulation and reveal various factors that affect the influence spreading. In Section III, we consider a more realistic social network which has high clustering coefficient and extend the local mean field model to analyze it. In Section IV, we generalize the three influence models to deterministic model and probabilistic model. In Section V, we extend our framework to allow multiple levels of rating on a given product. Related work is given in Section VI and Section VII concludes the paper.

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II. Basic Model

In this section, we study the influence spreading in OSNs. As stated before, users who have purchased a product can also influence their neighbors or friends. Our goal is to study what is the fraction of users that will purchase the product at the steady state? It is important to point out that the outcome depends on how users influence each other. In the following, we consider three influence mechanisms in OSNs.

A. Modeling Social Networks as Scale-free Random Graphs

For simplicity of presentation, we model the underlying OSN as an *infinite scale-free* [2] sparse "random" graph $G(V,E)$. By "infinite sparse random graph", we mean the limit of a series of graphs (see [3] Sec. 3.3), that is, $G^{(n)}: n \to \infty$ where $G^{(n)}$ is an n-node random graph with fixed degree distribution. In later section, we also extend the model for graphs with high clustering coefficient. In scale free graphs, the fraction of nodes that have k neighbors, denoted by $P_0(k)$, is proportional to $k^{-\gamma}$ for large values of k, or $P_0(k) \propto k^{-\gamma}$, where γ is a positive constant value[1]. Note that for a realistic OSN (e.g., Facebook), the number of users is in the order of 10^6 or larger, thus the infinity assumption is justified.

Each user is represented as a node in $G(V,E)$, and if node i decides to purchase a product, it may also influence its neighbors to purchase. The tight dependency among nodes makes the analysis difficult, e.g., if nodes a and b have a common neighbor, say node c, then the purchasing behaviors of nodes a and b are not independent with each other as node c may affect their decisions. In general, dependency may occur even if nodes are multiple links away from each other. This type of multi-nodes interaction is generally difficult to solve exactly due to the combinatorics generated by the interactions when summing over all possible influences.

To overcome this problem, we decouple the influence for each node. Consider a node i in graph $G(V,E)$, all influence to it comes from its neighbors or component C_i. To compute the total influence to node i, we can reshape C_i into a tree-like graph T_i rooted at i and consider influence algorithms on this transformed graph. We call T_i the *local field* of node i. To compute the average influence on an arbitrary node, one direct approach is to estimate the influence spreading on all $|V|$ local fields, and then take their average. Another approach, which is what we take, is to use a random graph to represent the mean of all $|V|$ local fields, then estimate the influence spreading only on this random graph. This random graph has probability $1/|V|$ to take shape T_i, and we call it the local mean field (LMF) of G. The construction of LMF of G can be described as follows. We model it as a tree with root node r which has $\deg(r)$ neighbors, say $v_1, v_2, \ldots, v_{\deg(r)}$, where $\deg(r)$ follows the same distribution with graph G, i.e., $P_0(k)$, and all other nodes follow the degree distribution of $\frac{kP_0(k)}{\sum kP_0(k)}$. Based on the above construction, the influence spreading in G can be approximated by the influence spreading in the LMF, and the accuracy is guaranteed by the following proposition.

Proposition 1: Let G be an infinite sparse random graph with asymptotic degree distribution, then for an arbitrary node r, the local topology of the component rooted at r can be modeled as a tree with high probability.

Remark: The local field of a node in an arbitrary finite graph usually contains loops and the local mean field can be very complex. However, for an infinite sparse random graph, the local fields can be approximated as trees with high probability. This result can be proven by using local weak convergence [4], and Lelarge et al. also show such approximation in [5], [6]. The basic idea is that, for an infinite sparse random graph G, the exploration of the successive neighborhoods of a given vertex can be approximated by a Galton-Watson branching process [7] as long as the exploration is local. Intuitively, this is because for an infinite sparse random graph, there only exists a small number of cross edges, moreover, the influence spreading through the cross edges is negligible comparing to the total influence, so the local topology can be *approximated* as a tree. Due to a recursive tree structure, the influence to the root node r is *independent* between any two sub-trees. We can then easily analyze the overall influence by all nodes to the root r. Since any node in a given G can be chosen as the root, the performance measure (e.g., average influence from all nodes to the root) can be applied to any node in graph G. Therefore, the average influence spreading in G can be approximated by the average influence spreading in the LMF.

To construct the LMF rooted at node r, we first need to obtain the degree distributions of the root node r and its children nodes. For a LMF tree rooted at r, $\deg(r)$ follows the same power law distribution with G, or:

$$\text{Prob}[\deg(r)=k] = P_0(k) = \frac{k^{-\gamma}}{\zeta(\gamma)}, \quad k=1,2,\ldots \tag{1}$$

where $\zeta(\gamma)=\sum_{k=1}^{\infty} k^{-\gamma}$ is the Riemann zeta function. We can also derive the degree distribution of any descendant node of r. The result is summarized in the following lemma.

Lemma 1: For an infinite random power law graph, the probability that a descendant node has degree k is:

$$P_1(k) = \frac{k^{1-\gamma}}{\zeta(\gamma-1)}, \quad \text{for } k=1,2,\ldots \tag{2}$$

Proof: Consider a descendent node b whose parent is node a, which we denote as $a \sim b$, we have:

$$P_1(k) = \text{Prob}[\deg(b)=k|a\sim b] = \frac{\text{Prob}[\deg(b)=k, a\sim b]}{\text{Prob}[a\sim b]} = \frac{\text{Prob}[\deg(b)=k, a\sim b]}{\sum_{k=0}^{\infty}\text{Prob}[\deg(b)=k, a\sim b]}. \tag{3}$$

To determine $\text{Prob}[\deg(b)=k, a\sim b]$, we will take $G(V,E)$ as $G^{(n+1)}$ and let $n\to\infty$. We use the fact that $G^{(n+1)}$ is a random graph, which means that the neighbors of a given node v are evenly distributed over $V-\{v\}$. Since $n=|V|-1$, then for node b, it has C_n^k ways to choose k neighbors, and C_{n-1}^{k-1} of them choose a as a neighbor, hence:

$$\frac{\text{Prob}[\deg(b)=k, a\sim b]}{\text{Prob}[\deg(b)=k]} = \frac{C_{n-1}^{k-1}}{C_n^k} = \frac{k}{n}. \tag{4}$$

Substitute (1) and (4) into (3), we have:

$$P_1(k) = \frac{\text{Prob}[\deg(b)=k]\frac{k}{n}}{\sum_{k=0}^{\infty}\text{Prob}[\deg(b)=k]\frac{k}{n}} = \frac{k^{1-\gamma}}{\zeta(\gamma-1)},\ k=1,2,\ldots$$

[1] The typical value of γ is in the range of $2<\gamma<3$.

Thus, the degree distributions of the descendants of node r all follow a *shifted power-law distribution* $P_1(k)$. ■

Now the local mean field is completely determined, and in the following sections, we will use it to study several influence mechanisms.

B. q-Influence Model

Suppose a company provides free samples as advertisement to $\rho < 1$ fraction of users. Users receiving the free sample will buy the product by their own will with probability p^+, while users who do not receive the free sample may also buy the product by their own will with probability p^-. We assume $p^+ > p^-$. Users who buy the product can also influence their friends (e.g., neighbors in the social network) to purchase with probability q. Our goal is to derive the fraction of users that will eventually purchase the product.

To address the above problem, let us first define some random variables. Let ϕ_i be the Bernoulli random variable indicating whether node i decides to purchase the product by his own will (e.g., without the influence of other nodes), then ϕ_i has the parameter μ where

$$\mu = \rho p^+ + (1-\rho)p^-. \tag{5}$$

Let θ_{ij} be the Bernoulli random variable indicating whether node i can influence his neighbor j to purchase the product. Under the $q-$influence model, if node i buys the product, then node i can make his neighbor j also purchase the product with probability q. Clearly, we have $P\{\theta_{ij} = 1\} = q$.

Let us first illustrate how nodes can influence other nodes via a *deterministic* example. Consider a *finite* tree with a pre-defined root r and all known variables ϕ_i and θ_{ij}. For node i, if ϕ_i is 1, obviously, it buys the product; if i has a neighbor j such that $\phi_j = \theta_{ji} = 1$, i will also buy the product. If neither of these two conditions hold, i may still buy the product if there is a path $i - i_1 - i_2 \ldots - i_k$ such that $\phi_{i_k} = \theta_{i_k i_{k-1}} = \ldots = \theta_{i_1 i} = 1$. Otherwise, i will not buy the product. Therefore, to compute the final state of the root node r (i.e., whether node r will purchase the product or not), we can update the states of all other nodes in this tree in a bottom-up manner. That is, we can determine the state ϕ_i of any leaf node i. Given the values of ϕ_i in the leaf nodes, we can determine the state of their parent nodes based on the influence model. It is important for us to point out that in this derivation, we only care about the final state of the root node. Moreover, we can ignore the influence propagating down the tree, i.e., the influence from the root node to the leaf node. Specifically, for node i and its child node j, we only need to count the influence from node j to node i, and we can ignore the influence from node i to node j. The rationale is that if the influence from j to i comes from j's parent node i, but not its child nodes, then it implies that node i has already purchased the product, i.e., $\phi_i = 1$. Obviously, the influence from j to i gives no contribution to node i so it can be ignored. This fact will be shown by the equivalence of the Monte Carlo algorithm and the enhanced algorithm in Section IV-A.

We generalize the above intuition in an infinite-depth random tree. Let X indicate whether the root node r finally buys the product, $\text{cld}(a)$ be the set of children of node a, Y_i indicate whether a non-root node i buys the product only due to the influence of the advertisement and its descendants, then $X = Y_r$. Based on the definition of the q-influence model, we have the following relationships:

$$1 - Y_i = (1-\phi_i) \prod_{j \in \text{cld}(i)} (1 - \theta_{ji} Y_j) \tag{6}$$

$$1 - X = (1-\phi_r) \prod_{j \in \text{cld}(r)} (1 - \theta_{jr} Y_j). \tag{7}$$

In effect, Y_i sums up all influence from all descendant nodes of node i, and X sums up all influence from the subtrees. Consider X, Y_i as Bernoulli random variables with mean $E[X]$, $E[Y_i]$, then we can prove that (6) and (7) have a desired solution, and $E[X]$ is the final fraction of buyers. To derive $E[X]$, we first present the following theorem.

Theorem 1: For the infinite local mean field tree, all $Y_i, i \neq r$ are identically distributed. If Y_i and Y_j are at the same depth, then they are also independent of each other.

Proof: Intuitively, since the local topology can be modeled as a tree, we only consider the influence that comes from the subtrees, if Y_i and Y_j are at the same depth, they must be independent. On the other hand, since we consider the local mean field tree, which is the average of all local trees, all $Y_i, i \neq r$ must be identically distributed. For the reason why infinite sparse random graphs and the associated local fields can be treated as trees, please refer to [3], [4]. ■

By Theorem 1, we can let $Y_j \sim Y$ for all $j \neq r$. To solve (6) and (7), we take expectation on both sides and we have:

$$1 - E[Y_i] = (1-\mu) E\left[\prod_{j \in \text{cld}(i)} (1 - \theta_{ji} Y_j)\right]$$

$$1 - E[X] = (1-\mu) E\left[\prod_{j \in \text{cld}(r)} (1 - \theta_{jr} Y_j)\right].$$

To derive the expectation term on the right hand side, note that θ_{ji} and Y_j that share the same parent are all independent of each other, and we condition on the node degree:

$$E\left[\prod_{j \in \text{cld}(i)} (1 - \theta_{ji} Y_j)\right] = \sum_{k=0}^{\infty} P_1(k+1) \prod_{j=i_1}^{i_k} E[1 - \theta_{ji} Y_j] = \sum_{k=0}^{\infty} P_1(k+1)(1 - qE[Y])^k \tag{8}$$

$$E\left[\prod_{j \in \text{cld}(r)} (1 - \theta_{jr} Y_j)\right] = \sum_{k=1}^{\infty} P_0(k) \prod_{j=r_1}^{r_k} E[1 - \theta_{jr} Y_j] = \sum_{k=1}^{\infty} P_0(k)(1 - qE[Y])^k. \tag{9}$$

Here i_j is the j^{th} child of node i, and we use Theorem 1 in (8) and (9). $P_1(k)$ is the probability that a descendant node has degree k and $P_0(k)$ is the probability that the root node has degree k. We finally obtain the *recursive distributional*

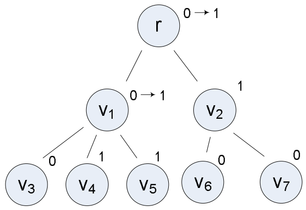


Fig. 1: Deterministic example for $m-$threshold model, $m = 2$.

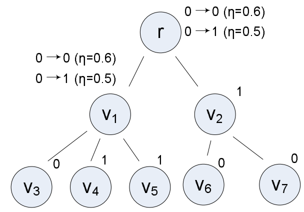


Fig. 2: Deterministic example for majority rule model.

equation (RDE) for the q-influence model:

$$1 - E[Y] = (1-\mu)\sum_{k=0}^{\infty} P_1(k+1)(1-qE[Y])^k \quad (10)$$

$$1 - E[X] = (1-\mu)\sum_{k=1}^{\infty} P_0(k)(1-qE[Y])^k. \quad (11)$$

The performance measure, $E[X]$, is the fraction of users that will eventually purchase the product. One can easily show that the solution of the above equations does exist and the valid value of $E[X]$ is the smallest non-negative solution of (10) and (11). Moreover, this value can be easily computed by standard iterative method. We will prove a generalization of this claim in Section IV-B.

C. m-Threshold Influence Model

In the $m-$threshold influence model, a user will buy the product either by his own will, or when at least m of his friends (or neighbors) have purchased the product. To illustrate, consider a deterministic example on a finite tree in Fig. 1. As before, let the random variable $\phi_i = 1$ if node i decides to purchase by its own will and $\phi_i = 0$ otherwise. In this deterministic example, the value of ϕ_i is shown and labeled in the figure. Suppose we set the threshold $m = 2$, then node v_1 will buy the product under the influence of node v_4 and v_5. Also, the root node r will buy the product under the influence of node v_1 and v_2. In general, to compute the state of the root node under the $m-$threshold influence model, we can apply the same bottom-up updating algorithm.

We still employ the same notations X, cld(a) and Y_i to represent the same meanings as before, and we have $Y_i \sim Y$ for all $i \neq r$. By the definition of the m-threshold influence model, we have the following relationships:

$$1 - Y_i = (1-\phi_i)\cdot \mathbf{1}\left[\sum_{j\in \mathrm{cld}(i)} Y_j < m\right] \quad (12)$$

$$1 - X = (1-\phi_r)\cdot \mathbf{1}\left[\sum_{j\in \mathrm{cld}(r)} Y_j < m\right]. \quad (13)$$

Here the Bernoulli random variable $\mathbf{1}[\sum_{j\in \mathrm{cld}(i)} Y_j < m]$ indicates whether less than m friends of node i have contributed influence to i. Local mean field method can also be applied to (12) and (13) so as to compute the state distribution of the randomly chosen root node. Taking expectation on both sides of (12) and (13), we have:

$$1 - E[Y_i] = (1-\mu)\mathrm{Prob}\left[\sum_{j\in \mathrm{cld}(i)} Y_j < m\right]$$

$$1 - E[X] = (1-\mu)\mathrm{Prob}\left[\sum_{j\in \mathrm{cld}(r)} Y_j < m\right].$$

To derive the probability term on the right side of the above equations, we can condition on the number of children nodes:

$$\mathrm{Prob}\left[\sum_{j\in \mathrm{cld}(i)} Y_j < m\right] = \sum_{k=0}^{\infty} P_1(k+1)\sum_{j=0}^{\min\{m-1,k\}} C_k^j E[Y]^j (1-E[Y])^{k-j}$$

$$\mathrm{Prob}\left[\sum_{j\in \mathrm{cld}(r)} Y_j < m\right] = \sum_{k=1}^{\infty} P_0(k)\sum_{j=0}^{\min\{m-1,k\}} C_k^j E[Y]^j (1-E[Y])^{k-j}.$$

So the final recursive distributional equation (RDE) for the $m-$threshold mechanism is:

$$1 - E[Y] = (1-\mu)\sum_{k=0}^{\infty}\sum_{j=0}^{\min\{m-1,k\}} P_1(k+1) C_k^j E[Y]^j (1-E[Y])^{k-j} \quad (14)$$

$$1 - E[X] = (1-\mu)\sum_{k=1}^{\infty}\sum_{j=0}^{\min\{m-1,k\}} P_0(k) C_k^j E[Y]^j (1-E[Y])^{k-j}. \quad (15)$$

In other words, $E[X]$ is the fraction of users in the social network that will eventually purchase the product. Again, the above equations have solution(s) and the smallest non-negative one gives the valid value of $E[X]$.

D. Majority Rule Influence Model

In the majority rule influence model, a user will buy the product either by his own will, or if over η fraction of his friends have bought the product. Fig. 2 shows a deterministic example of a finite tree under the majority rule model. As before, ϕ_i indicates whether node i decides to purchase by its own will. In this deterministic example, the values of ϕ_i are all known and labeled in the figure. If we define the majority as 50%, according to the initial condition, v_1 will be influenced to purchase the product since half of his friends, v_4 and v_5, have bought the product. Also, the root node will be influenced to purchase the product as half of its friends (e.g., v_1) have purchased the product.

According to the definition of the majority rule influence model, and by using the same notations as before, the following equations hold for the majority rule influence model:

$$1 - Y_i = (1-\phi_i)\cdot \mathbf{1}\left[\sum_{j\in \mathbf{cld}(i)} Y_j < \eta \deg(i)\right] \quad (16)$$

$$1 - X = (1-\phi_r)\cdot \mathbf{1}\left[\sum_{j\in \mathbf{cld}(r)} Y_j < \eta \deg(r)\right]. \quad (17)$$

Here the Bernoulli random variable $\mathbf{1}[\sum_{j\in\mathbf{cld}(i)} Y_j < \eta \deg(i)]$ indicates whether the fraction of i's friends that have purchased the product is less than the majority line η. In the example of Fig. 2, if we raise the majority value η from 50% to 60%, then v_1 and r will not be influenced anymore. To see this, when two of the four neighbors of v_1, v_3 and v_4 are activated, so $\eta = 0.5$ and this activates v_1 but when $\eta = 0.6$, no activation will occur. To solve X, we take expectation on both sides of (16) and (17):

$$1 - E[Y_i] = (1-\mu)\text{Prob}\left[\sum_{j\in \mathbf{cld}(i)} Y_j < \eta \deg(i)\right]$$

$$1 - E[X] = (1-\mu)\text{Prob}\left[\sum_{j\in \mathbf{cld}(r)} Y_j < \eta \deg(r)\right].$$

To derive the probability term on the right side, we condition on the number of children:

$$\text{Prob}\left[\sum_{j\in \mathbf{cld}(i)} Y_j < \eta \deg(i)\right] = \sum_{k=0}^{\infty} P_1(k+1) \sum_{j=0}^{\lceil \eta(k+1)\rceil -1} C_k^j E[Y]^j (1-E[Y])^{k-j}$$

$$\text{Prob}\left[\sum_{j\in \mathbf{cld}(r)} Y_j < \eta \deg(r)\right] = \sum_{k=1}^{\infty} P_0(k) \sum_{j=0}^{\lceil \eta k\rceil -1} C_k^j E[Y]^j (1-E[Y])^{k-j}.$$

The final recursive distributional equation (RDE) for the majority rule influence model is:

$$1-E[Y] = (1-\mu)\sum_{k=0}^{\infty}\sum_{j=0}^{\lceil \eta(k+1)\rceil -1} P_1(k+1) C_k^j E[Y]^j (1-E[Y])^{k-j} \quad (18)$$

$$1-E[X] = (1-\mu)\sum_{k=1}^{\infty}\sum_{j=0}^{\lceil \eta k\rceil -1} P_0(k) C_k^j E[Y]^j (1-E[Y])^{k-j}. \quad (19)$$

Again, $E[X]$ is the final fraction of buyers.

Intuitively, the three influence models may play different role under different situations, i.e., some influence model may be better than the others in certain cases. In particular, the $q-$influence model exhibits more independence since one node can influence the purchase behavior of another node, or a node's purchase behavior can be *independently* influenced by all his neighbors. On the other hand, the $m-$threshold model and the majority model allow more *dependency*: one is only influenced when more than a specific number of his neighbors decide to do the purchase. In other words, these two models allow one to capture more *network externality effect*.

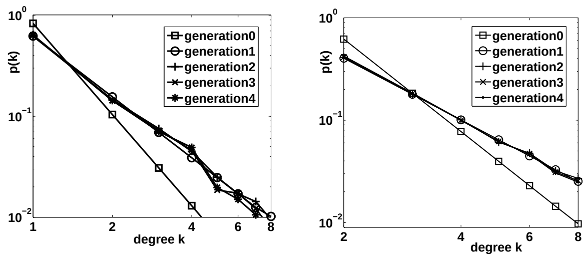
E. Performance Evaluation

In this subsection, we will evaluate the performance of the influence models presented in previous subsections. The local mean field model assumes an infinite scale-free random graph *without* self-loops and duplicate edges. To evaluate its effectiveness, we first need to generate a random graph with sufficiently large number of nodes so as to approximate the infinite condition. Here we propose a technique that can generate a finite random graph (without self-loops or duplicate edges) for any degree distribution.

Our approach has two steps: degree allocation and randomization. In the first step, we generate $|V|$ numbers according to a given distribution. If the sum is not even, discard them and generate again. Otherwise, assign one number to one node and take it as its degree. We order the nodes from the highest degree assignment to the lowest and add edges to them one by one. For node v with degree assignment n_v, we randomly choose n_v nodes that have not reached their assigned degree and connect them to v. Repeat the above process until every node has reached its assigned degree. In the randomization step, we randomly shuffle two edges for $5D$ rounds where D is the total degree of the graph. In each round, we randomly choose two edges (a, b) and (c, d), and shuffle them into $(a, d), (c, b)$ or $(a, c), (b, d)$. Obviously, such operation does not change the total degree of any node or the degree distribution of the whole graph.

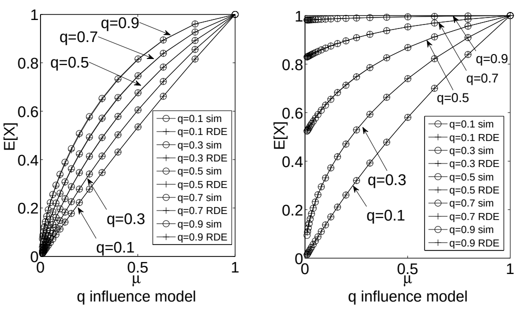
In general, a node v is in generation i of node r if v has distance i to r. As we see from Lemma 1, the degree distribution P_1 of the first generation may be different from the original degree distribution P_0 of graph G. Fig. 3 shows the degree distribution of the first five generations for two power law random graphs. Both graphs have 10000 nodes and exponent $\gamma = 3$. Fig. 3(a) shows the distributions of graph with minimum degree of one and average degree of 1.37. Fig. 3(b) has minimum degree of two and average degree of 3.19. Both graphs are randomized. From Fig. 3(a), one can observe that (a) nodes in generation 1 to generation 4 all follow the *same* shifted power law distribution $P_1(k)$ according to Lemma 1, (b) both graphs have a dominating fraction of minimum degree nodes. In the following, we validate our theoretical results of the three influence models using the two power law random graphs in Fig. 3. In each simulation, we generate 10 graph instances with 10000 nodes and for each graph instance, we simulate the influence spreading process and compute the final fraction of buyers for 1000 times. Then we take the average of all the 10000 values as $E[X]$.

1) q-Influence Model: Fig. 4 shows the simulation as well as theoretical results by (10) and (11). The horizontal axis μ (or from (5)) is the initial fraction of users that purchase



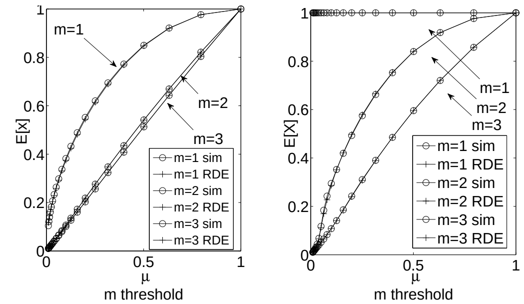
(a) (b)

Fig. 3: The degree distribution by generations of two power law random graphs. (a) Power law random graph with minimum degree of one. (b) Power law random graph with minimum degree of two.



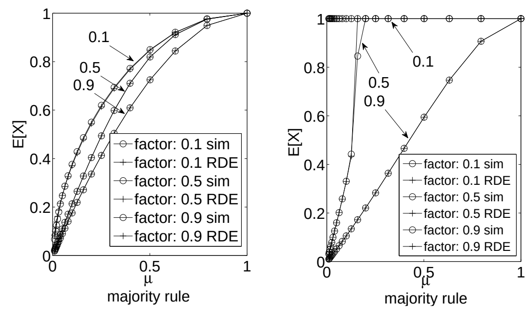
(a) (b)

Fig. 4: Impact of q influence model in random scale-free graph. (a) Graph with minimum degree one. (b) Graph with minimum degree two.



(a) (b)

Fig. 5: Impact of $m-$threshold influence model in random scale-free graph. (a) Graph with minimum degree one. (b) Graph with minimum degree two.



(a) (b)

Fig. 6: Impact of majority rule influence model in random scale-free graph. (a) Graph with minimum degree one. (b) Graph with minimum degree two.

the product, and the vertical axis $E[X]$ is the final fraction of buyers. In each figure, there are five simulation curves and five theoretical curves corresponding to different q values from 0.1 to 0.9. First of all, we can see that the theoretic results fit very well with the simulation results. In both figures, when q is very small, say 0.1, $E[X]$ almost equals μ, which means that mutual influence among users is very weak. When q becomes large, the fraction of final buyers rises up quickly as μ increases, then they saturate the network and eventually every node will purchase the product. We also observe that $E[X]$ is much higher in Fig. 4(b) because the underlying graph has a higher average degree, which indicates that the more connected network is easier for the influence to diffuse. We observe the outbreak behavior in high degree graph, i.e., even if an infinitesimal value of μ ($\mu \approx 0$) can still entice a large fraction of users to purchase (large value of E[X]). This observation agrees with the result in [8] that the percolation threshold for scale-free graph is zero.

2) m-Threshold Influence Model: Fig. 5 shows the simulation as well as theoretical results by (14) and (15). In each figure, there are three simulation curves and three theoretical curves corresponding to different m values from 1 to 3. In both graphs, when $m = 3$, $E[X]$ almost equals μ, which means that mutual influence among users is very weak. This is because in both figures, most nodes have degree of one or two, so the threshold $m = 3$ is rarely reached. For fixed m, the curve is higher in Fig. 5(b) where the power law graph has a higher average degree. In Fig. 5(a), about 80% nodes have degree one, threshold $m = 2$ is still too high for influence to diffuse. When $m = 1$, the m-threshold model is equivalent with q-influence model where $q = 1$. The jump of $E[X]$ from 0 to about 0.1 near $\mu = 0$ suggests that the giant component in the low degree scale-free graph takes up about 10% of the total population. In Fig. 5(b) where over 60% of nodes have degree two, a threshold that is higher than two severely restricts the influence diffusion among users. However, we also observe the outbreak behavior when $m = 1$.

3) Majority Rule Influence Model: Fig. 6 shows the simulation as well as theoretical results by (18) and (19). In each figure, there are three simulation curves and three theoretical curves corresponding to majority factor η of 10%, 50%, and 90%. We can see that the theoretic results fit very well with the simulation results. In both figures, even when $\eta = 0.9$, the curve is obviously above the 45 degree line, which means that there is still considerable mutual influence in the network. In the low degree graph (Fig. 6(a)), the three curves are very close to each other while in the high degree graph (Fig. 6(b)), they are widely separated. While the $\eta = 0.5$ and $\eta = 0.1$ curves are higher in the high degree graph than in the low degree graph, $\eta = 0.9$ curve is lower in the high degree graph. Again, we observe outbreak behavior in Fig. 6(b).

To show the efficiency of our model, we compare the time cost of running the simulation and solving the model (See Table I). In Table I, the parameters for simulation are as follows: $|V| = 10000, |E| \approx 24000, \mu = 0.05$, the minimum degree of the graph is two. For each simulation, we run 10 graph instances and 1000 times for each instance and take the

TABLE I: Computational overheads.

cases		time (ms)
q-model $q=0.3$	simulation	48348
	theory	31
m-threshold $m=5$	simulation	43657
	theory	32
majority rule $\eta=0.7$	simulation	44969
	theory	41

average value. It is important for us to state that computing $E[X]$ by our mathematical model is more than 1000 times faster than simulation for all influence models. Moreover, if the graph size is very large, then estimating the influence via simulation becomes more computationally expensive, while using our model has the clear computational advantage.

In summary, the RDE framework is very accurate for scale-free random graphs under all influence models we have studied. Since the outbreak behavior appears in all cases, to design efficient advertisement strategies, a company may offer a little more free samples so as to trigger the outbreak behavior, which can help the company to achieve a higher revenue. This short of free samples allocation will he useful for OSNs wherein users interact frequently, e.g., via the Power Tweet in Twitter. Here we only consider the scale-free characteristic of OSNs and derive various implications. In the next section, we will take into account another important feature of OSNs to make our model more realistic.

III. Scale-free Graph with High Clustering Coefficient

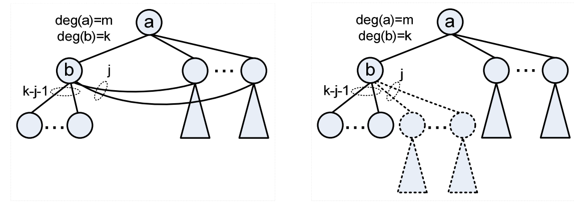
In OSNs, two common friends of a user are usually friends of each other. This implies that graphs of OSNs usually exhibit high *clustering*. In this section, we extend our methodology to analyze scale-free graphs with high clustering property. We use the definition which was first proposed by Watts and Strogatz in [9] to characterize the clustering coefficient.

Definition 1: The clustering coefficient of a graph $G(V,E)$ is

$$c = \frac{1}{|V|}\sum_{v\in V}\frac{t_v}{k_v(k_v-1)/2} \tag{20}$$

where k_v is the degree of node v, and t_v is the number of edges in the neighborhood of node v. For the above definition, we ignore nodes with degree of one.

Obviously, the scale-free graph with high clustering coefficient cannot be modeled as a tree. For infinite scale-free graph, the depth of a local tree is infinite. So the influence that comes from the neighbors in the same generation can be approximated by the influence that comes from the children in the next generation. We call the edge between sibling nodes as the *cross edge*. For example, in Fig. 7(a), node b connects with j children of a's. To approximate the influence to b, these j siblings can be taken as b's children. Based on this fact, we construct another new tree, the local mean field tree, to approximate the influence spreading in the original graph. The way of constructing such new tree is shown in Fig. 7(b). To compute the degree distribution of descendent



(a) (b)

Fig. 7: Graph with high clustering coefficient and its tree approximation. (a) Graph with high clustering coefficient. (b) LMF tree approximation for node b.

nodes in the local mean field tree, we first need to determine the probability that a cross edge appears. Since the local mean field is the average of all local trees, when we construct the local mean field, we can use the clustering coefficient of the graph to represent a random node's clustering coefficient. In other words, one can assume that all cross edges appear with probability c. We use this approximation to compute the degree distribution of descendent nodes in the corresponding LMF tree and analyze the influence spreading in the new tree structure. In essence, due to the cross edges, the degree distribution of the descendant nodes in the LMF tree does not follow shifted power law any more, and we can compute it as follows. Consider a descendant node b whose parent is node a (see Fig. 7(b)), we have:

$$\begin{aligned}
&P(deg(b)=k|deg(a)=m)\\
&=\sum_{j=0}^{\min(k-1,m-1)} p(b\overset{j}{\sim}a)\cdot p(deg(b)=k|deg(a)=m, b\overset{j}{\sim}a)\\
&=\sum_{j=0}^{\min(k-1,m-1)}\binom{m-1}{j}c^j(1-c)^{(m-1-j)}\frac{p_0(k)\frac{\binom{k-j-1}{n-j-1}}{\binom{k}{n}}}{\sum\limits_{k=1}^{\infty}p_0(k)\frac{\binom{k-j-1}{n-j-1}}{\binom{k}{n}}}\\
&=\sum_{j=0}^{m-1}\binom{m-1}{j}c^j(1-c)^{(m-1-j)}\frac{k(k-1)...(k-j)k^{-\gamma}}{\sum\limits_{k=1}^{\infty}k(k-1)...(k-j)k^{-\gamma}}
\end{aligned}$$

where $b\overset{j}{\sim}a$ means b connects j edges with the children of node a. We summarize the results in the following lemma.

Lemma 2: For an infinite scale free graph with clustering coefficient c, the probability that a descendant node in the LMF tree has degree k is:

$$\begin{aligned}
P_1(k) = &\sum_{m=1}^{\infty}\frac{m^{-\gamma}}{\zeta(\gamma)}\sum_{j=0}^{m-1}\binom{m-1}{j}c^j(1-c)^{(m-1-j)}\\
&\times\frac{k(k-1)\cdots(k-j)k^{-\gamma}}{\sum\limits_{k=1}^{\infty}k(k-1)\cdots(k-j)k^{-\gamma}},\ k=1,2\ldots.
\end{aligned}\tag{21}$$

Note that when the clustering coefficient $c=0$, (21) becomes the shifted power law. Now the local mean field of scale-free graph with high clustering coefficient is completely determined. One can use the LMF model defined in Section II-B (10) and (11), Section II-C (14) and (15), and Section II-D (18) and (19) to analyze the influence spreading in infinite scale-free, high clustering graph by using (21) to substitute $P_1(k)$.

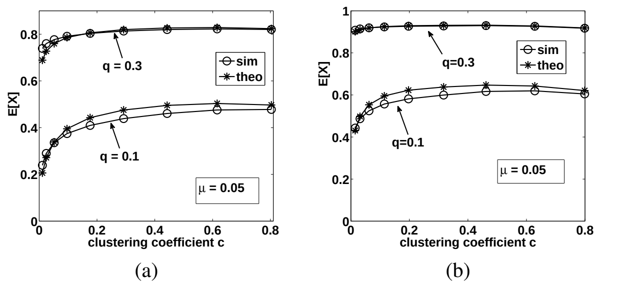


Fig. 8: Impact of the $q-$influence model in scale-free graph with high clustering coefficient. (a) Minimum degree 3. (b) Minimum degree 5.

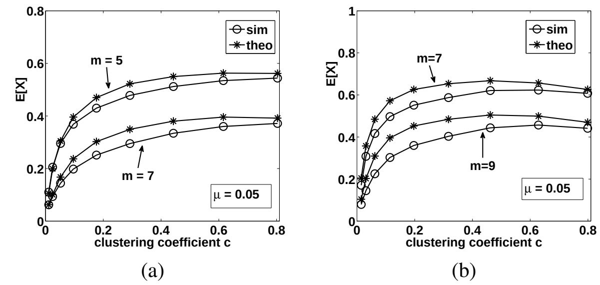


Fig. 9: Impact of the $m-$threshold influence model in scale-free graph with high clustering coefficient. (a) Minimum degree 3. (b) Minimum degree 5.

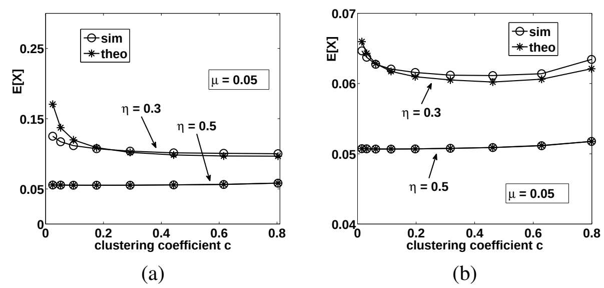


Fig. 10: Impact of the majority rule influence model in scale-free graph with high clustering coefficient. (a) Minimum degree 3. (b) Minimum degree 5.

A. Performance Evaluation

In this subsection, we present the performance evaluation of the extended models presented above. Note that there is a body of work on topology generation, see, e.g. [10], [11]. In [10], the authors propose a method which is based on preferential attachment to generate scale free graphs with high clustering coefficient. In [11], the authors propose using Kronecker graphs which naturally obey common network properties to model networks. In this paper, we use the GLP model presented in [10] to generate a scale-free graph with high clustering coefficient, then run simulation in the generated graph to validate our model. To further show the accuracy of our model, we also run simulation in realistic social networks. We are interested in the relationship of the performance measure, $E[X]$, and the clustering coefficient c. In our simulation, we generate two types of scale-free graphs which have a minimum degree of 3 and a minimum degree of 5 respectively. Since ρ, the probability of a randomly chosen node will receive the free sample, is usually small, we fix $\mu = 0.05$.

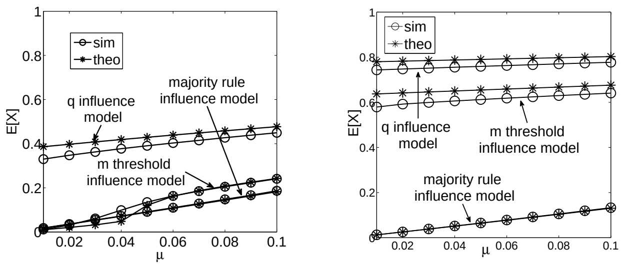
1) Performance of the q-Influence Model: Fig. 8 shows the simulation as well as theoretical results by the RDE, (10) and (11), which uses $P_1(k)$ presented in (21). The horizontal axis c is the clustering coefficient of the graph, and the vertical axis $E[X]$ is the final fraction of users that will eventually buy the product. In each figure, there are two sets of curves corresponding to different q values from 0.1 to 0.3. Firstly, we can see that the theoretic results fit well with the simulation results. We also observe that $E[X]$ is much higher in Fig. 8(b) because the underlying graph has a higher average degree. This indicates that high degree networks, which are more connected, are easier for the influence to diffuse. Moreover, we can see that when the clustering coefficient increases, the influence $E[X]$ also increases, but slower for high clustering coefficients. At last, we see that even an infinitesimal advertisement ($\mu = 0.05$) can still lead to high $E[X]$.

2) Performance of the m-Threshold Influence Model: Fig. 9 shows the simulation as well as theoretical results by the RDE, (14) and (15), but use $P_1(k)$ presented in (21). In each figure, there are two sets of curves corresponding to different m values. We can see that the theoretic results correctly characterize the shape of the simulation results curve. For fixed m ($m = 7$), the curve is higher in Fig. 9(b) where the power law graph has a higher average degree. Again, we can see that the influence measure $E[X]$ increases with c. Similar with the q influence model, we also observe that small μ can lead to high $E[X]$.

3) Performance of the Majority Rule Influence Model: Fig. 10 shows the simulation as well as theoretical results by the RDE, (18) and (19) that uses $P_1(k)$ presented in (21). In each figure, there are two sets of curves corresponding to majority factor η of 30%, 50%. We can see that the curves of the theoretic results and simulation results have the same tendency of variation. In both figures, when $\eta = 0.5$, $E[X]$ almost equals to μ, which means that the mutual influence among users is very weak. Again, we observe even low μ can still lead to high $E[X]$ when $\eta = 0.3$. Unlike the q-model and m threshold, when the clustering coefficient c increases, the final fraction of buyers $E[X]$ decreases. And $E[X]$ almost keeps the same for high clustering coefficient c.

In summary, inspired by the local mean field model for scale-free random graph (without high clustering coefficient), we extend the model for scale-free graph with high clustering coefficients, and the extended model can accurately predict the influence spreading in online social networks.

4) Model Validation on Realistic Networks: To further show the accuracy of our extended model, we also provide the performance evaluation on two realistic social networks with different clustering coefficients which are Arxiv GR-QC (General Relativity and Quantum Cosmology) collaboration network [12] and Arxiv AstroPh (Astro Physics) collaboration network [13] and the clustering coefficients are 0.53 and 0.63 respectively. Fig. 11(a) shows the results of Arxiv GR-QC collaboration network and Fig. 11(b) corresponds to the case of Arxiv AstroPh collaboration network. In both figures,



(a) (b)

Fig. 11: Performance evaluation on real social networks. (a) Arxiv GR-QC collaboration network where $c = 0.53$. (b) Arxiv AstroPh collaboration network where $c = 0.63$.

the horizontal axis, μ, is the initial fraction of users who buy a product and the vertical axis is the final fraction of buyers, $E[X]$. For both cases, We plot the impact of q influence model, m-threshold influence model and majority rule influence model in the same figure, and for the case of Arxiv GR-QC network, the corresponding parameters are $q = 0.25$, $m = 4$ and $\eta = 0.5$, and for the case of Arxiv AstroPh network, the corresponding parameters are $q = 0.25$, $m = 5$ and $\eta = 0.5$. The results show that our model can indeed provide a good approximation even for real networks.

IV. Generalized Influence Models

In the following, we present two generalized influence models, namely, the deterministic influence model and the probabilistic influence model. They subsume the previously discussed influence models.

A. Deterministic Influence Model

In this subsection, we formalize the product advertising problem on general graphs and define a class of influence models characterized by a 0-1 valued *mutual influence function*. This influence model is a generalization of the m-threshold and majority rule influence model we discussed in Section II.

Given a graph $G(V, E)$, each node in V has two states, influenced or not influenced. We represent the node v's state by a 0-1 random variable X_v and $X_v = 1$ means v is influenced. Suppose there is a background source such that every node is influenced by it independently with probability μ. Every node can also be influenced by its neighbors according to the *mutual influence function* $f(n, k)$, which indicates whether a node will be influenced by its neighbors when k of its n neighbors are influenced. For example, $f(n, k) = \mathbf{1}[k \geq m]$ characterizes m-threshold influence model while $f(n, k) = \mathbf{1}[k \geq \eta n]$ describes the majority rule influence model. One may design more complex influence models, say $f(n, k) = \mathbf{1}[k > \sqrt{n}]$. We assume $f(n, k)$ is *monotone*, that is, $f(n, 0) \neq 1$ and $f(n, k) \leq f(n, k + 1)$. One interesting question is, given an influence model, what is the probability that a randomly chosen node is influenced?

To answer this question, we first start with a simple example. Consider the tree shown in Fig. 12, the probability $E[X_1]$ that node 1 is influenced can be determined using the Monte Carlo method as follows:

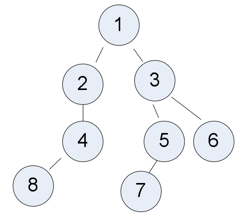


Fig. 12: Example of a tree for Monte Carlo Algorithm.

Monte Carlo Algorithm

1) $c \leftarrow 0$
2) repeat for N rounds where N is sufficiently large:
   a) initialize every node with state 0.
   b) assign every node state 1 with probability μ.
   c) repeat until no node changes state:
      - for every node, update its state according to the mutual influence function $f(n, k)$.
   d) if node 1 is influenced, $c \leftarrow c + 1$.
3) $E[X_1] \leftarrow c/N$.

The above algorithm follows the definition of $E[X_v]$ and determines the state of all nodes under each initial random assignment. Then it takes the frequency of node 1 being in state 1 as $E[X_1]$. One may wonder if it is necessary to determine the final state of all the other nodes in order to determine the state of the root node. The answer is no, and one can show that the following algorithm is *equivalent* to the Monte Carlo algorithm.

Enhanced Algorithm

1) $c \leftarrow 0$
2) repeat for N rounds where N is sufficiently large:
   a) initialize every node with state 0.
   b) construct the shortest path tree for node 1, m is the maximal depth.
   c) for $i = m, \ldots, 0$:
      - for every node in generation i, assign it state 1 with probability μ. Update its state according to the mutual influence function $f(n, k)$.
   d) If node 1 is influenced, $c \leftarrow c + 1$.
3) $E[X_1] \leftarrow c/N$.

Here we briefly explain why the enhanced algorithm is equivalent with Monte Carlo algorithm. Let state 1 be the influenced state and state 0 be the inactive state. Note that the shortest path tree of a node is actually the component that contains it. The two algorithms are almost the same except that in the enhanced version, when updating the state of non-root nodes, we do not count the influence from higher generations. Although such omission may cause the final state of some non-root nodes to have a lower value than that in the Monte Carlo algorithm, but it does not affect the final state of the root node. In case it does, then there must be an active node A propagating its influence down the tree and activate a descendant node B, and the activation of B should trigger another update up the tree to activate the root node. However, if A is already activated once, then the influence from B will have no impact since B can only influence the root by activating A in a tree topology. Therefore, we can safely ignore the influence from A to B when we only care about the state of the root node.

Let random variable Y_v be the state of node v after performing the enhanced algorithm for one round. Then Y_v is the accumulated influence at node v containing from its descendants. It is important to note that the enhanced algorithm can be formulated via the following iterative distributional equation:

$$1 - Y_i = (1 - \phi_i(\mu)) \left(1 - f\left(\deg(i), \sum_{j \in \mathrm{cld}(i)} Y_j\right)\right) \quad (22)$$

Here, $\phi_i(\mu)$ is Bernoulli random variables with mean μ. It means that a node is not influenced if and only if it is not influenced by the background and not influenced by its neighbors. Because we consider the infinite scale-free graph, for all non-root node i, Y_is are identically distributed, we denote their expectations as $E[Y]$. And we also let the Bernoulli random variable X indicate whether the root node buys the product. Taking expectation on both sides of (22), we have the following recursive distributional equations (RDE):

$$\begin{aligned} 1-E[Y] &= (1-\mu)\Big(1-\sum_{n=0}^{\infty} P_1(n+1) \\ &\quad \times \sum_{k=0}^{n} C_n^k f(n+1,k) E[Y]^k (1-E[Y])^{n-k}\Big) \quad (23) \\ 1-E[X] &= (1-\mu)\Big(1-\sum_{n=0}^{\infty} P_0(n) \\ &\quad \times \sum_{k=0}^{n} C_n^k f(n,k) E[Y]^k (1-E[Y])^{n-k}\Big). \quad (24) \end{aligned}$$

Here $P_1(n)$ is the degree distribution of descendant nodes and $P_0(n)$ is the degree distribution of the root. Equation (23) computes the accumulated influence along the shortest path tree in a bottom-up manner. And $E[X]$ is the expected probability that a randomly chosen root will purchase a product.

B. Probabilistic Influence Model

Under the probabilistic influence model, every node that purchases the product will influence, or activate its neighbors independently with probability q. The mutual influence function $f(n,k)$, is the probability a node will be influenced when k out of n neighbors are active. f is monotone if $f(n,0) \neq 1$ and $f(n,k) \leq f(n,k+1)$. Unless we state otherwise, we assume f to be monotonic function from now on. Here k is the number of neighbors that successfully contributes influence. The q-influence model we discussed in Section II is a special case of the probabilistic model with $f(n,k) = \mathbf{1}[k > 0]$. When $q = 1$ and f is 0-1 valued, probabilistic influence model becomes the deterministic model we discussed before.

As before, let Y_i be the influence accumulated at node i, θ_{ij} indicates whether a node i contributes influence to node j. Using similar argument in Section IV-A, the following relation holds for all $i \in V$:

$$1-Y_i = (1-\phi_i(\mu)) \left(1 - f\left(\deg(i), \sum_{j \in \mathrm{cld}(i)} \theta_{ji} Y_j\right)\right) \quad (25)$$

where $\phi_i(\mu)$ is Bernoulli random variable with mean μ. Equation (25) indicates that a node is not influenced if and only if it is not influenced by the background and not influenced by its neighbors. Again, for all non-root node i, $Y_i \sim Y$, and we still let X be the random variable that indicates whether the root node buys the product. Taking expectation on both sides of (25), we have the following recursive distributional equation (RDE):

$$\begin{aligned} 1-E[Y] &= (1-\mu)\Big(1-\sum_{n=0}^{\infty} P_1(n+1) \\ &\quad \times \sum_{k=0}^{n} C_n^k f(n+1,k) (qE[Y])^k (1-qE[Y])^{n-k}\Big) \quad (26) \\ 1-E[X] &= (1-\mu)\Big(1-\sum_{n=0}^{\infty} P_0(n) \\ &\quad \times \sum_{k=0}^{n} C_n^k f(n,k) (qE[Y])^k (1-qE[Y])^{n-k}\Big). \quad (27) \end{aligned}$$

Here $P_1(n)$ and $P_0(n)$ have the same meaning as before. $E[X]$ computed at the root in (27) is the probability that a randomly chosen node purchases the product, which equals to the expected fraction of users that will buy the product in the social network.

Denote $E[X], E[Y]$ as x, y respectively. Define function:

$$\begin{aligned} g(x) &= 1-(1-\mu) \\ &\quad \times \left(1-\sum_{n=0}^{\infty} P_1(n+1) \sum_{k=0}^{n} C_n^k f(n+1,k)(qx)^k (1-qx)^{n-k}\right) \\ h(x) &= 1-(1-\mu) \\ &\quad \times \left(1-\sum_{n=0}^{\infty} P_0(n) \sum_{k=0}^{n} C_n^k f(n,k)(qx)^k (1-qx)^{n-k}\right). \end{aligned}$$

Then (26) and (27) reduce to:

$$y = g(y) \quad x = h(y). \quad (28)$$

Let y_t be the average influence at time slot t and x_t be the fraction of influenced users at time slot t, then we have

$$x_0 = y_0 = \mu \quad y_{t+1} = g(y_t) \quad x_{t+1} = h(y_t).$$

By the definition, we have $E[X] = \lim_{t\to\infty} x_t$.

Theorem 2: Let $\mu, q \in [0,1]$, $f(n,k)$ be a monotone influence function, and $P_i(k)$ be an arbitrary distribution. Then the limit $\lim_{t\to\infty} x_t$ exists and it is the smallest non-negative solution of (26) and (27).

Proof: First of all, both $g(x)$ and $h(x)$ are non-decreasing. Consider the following weighted sum of $f(n,k)$ in $h(x)$:

$$\sum_{k=0}^{n} C_n^k f(n,k)(qx)^k (1-qx)^{n-k}$$

where the weight of $f(n,k)$ is the binomial term $C_n^k(qx)^k(1-qx)^{n-k}$. The total sum of the weights is $\sum_{k=0}^n C_n^k(qx)^k(1-qx)^{n-k}=1$, a constant. So increasing x is in effect shifting the weight from $f(n,k)$ of smaller k to those of larger k. Since $f(n,k)$ is monotone, it increases with k while n is fixed. Thus the weighted sum of $f(n,k)$ doesn't decrease with x, hence $h(x)$ is non-decreasing. Similar argument holds for $g(x)$. Secondly, it is easy to see that, for $x\in[0,q^{-1}]$, $y_1=g(\mu)\geq\mu=y_0$. Thus by the monotonicity of $g(x)$ and mathematical induction, $y_{t+1}>y_t$ for all $t\geq 0$. Hence the series $\{y_t\}$ is non-decreasing. On the other hand, $\{y_t\}$ is upper-bounded by 1, thus it must have a limit y^* and $g(y^*)=y^*$. We claim the y^* is the smallest non-negative solution of $g(y)=y$. Otherwise, suppose $0\leq y'<y^*$ and $y'=g(y')$. Apparently $y'\geq\mu$. By the monotonicity of $g(x)$, $g(\mu)=y_1\leq g(y')=y'<g(y^*)=y^*$. Apply it repeatedly, we have $y_t\leq y'<y^*$ for all $t>0$. Thus $\lim_{t\to\infty}y_t\leq y'<y^*$, a contradiction.

Since $\lim_{t\to\infty}y_t$ exists and it is the smallest non-negative solution of $y=g(y)$, the limit $\lim_{t\to\infty}x_t=h(\lim_{t\to\infty}y_t)$ also exists and is the smallest solution of (26) and (27). ■

V. Multi-state Model

In this section, we consider a more general advertising problem. We assume that every user can be influenced by the advertiser independently and has a personal rating about the product ranging from 0 to t, where a higher value of rating implies a higher preference on the product. The probability that the users would have rating i is p_i, which depends on the quality of the product and the advertising strategy of the company. We also assume p_is are all known quantities in our discussion. Without loss of generality, we assume every user provides a rating on the product. If a user does not provide a rating, we assume the user perceives the product with rating 0. Lastly, we define the concept of an *acceptance line*. An acceptance line a implies that a user will only buy the product if his/her rating of the product is above or equals to a.

Users can be influenced by the rating of their friends according to the mutual influence function $f_{rs}(n;k_0,k_1,k_2,\ldots,k_t)$, $0\leq r,s\leq t$. The mutual influence function defines the *rule* that a user updates its rating: if a node v with rating r has n friends, and k_i of its friends give rating i, then node v will change its rating to s with probability $f_{rs}(n;k_0,k_1,k_2,\ldots,k_t)$. For convenience, let the integer vector $\boldsymbol{k}=(k_0,k_1,k_2,\ldots,k_t)$ and we also write the mutual influence function as $f_{rs}(n;\boldsymbol{k})$. We define a partial order on $\mathbb{N}^t$ in which $(k_0,\ldots,k_i,\ldots,k_j,\ldots,k_t)<(k_0,\ldots,k_i-1,\ldots,k_j+1,\ldots,k_t)$. We say the social network is *stable* if no one can change its rating according to f_{rs}. A user will buy the product if when the social network is stable, its rating is not lower than the *acceptance line* a. Again, we want to find the fraction of users that will eventually buy the product.

Consider the following mutual influence functions:

$$f_{10}(n;k_0,k_1) = \mathbf{1}[k_0>n/2] \quad (29)$$
$$f_{01}(n,k_0,k_1) = \mathbf{1}[k_1>n/2]. \quad (30)$$

The above functions indicate a *strict majority rule*: a user will have rating one if the majority of his friends have rating one and it will have rating zero if the majority of his friends have rating zero. So if the acceptance line is $a=1$, then a user will buy the product only when the majority of its friends buy the product. Note that this model is different from the majority rule influence model we discussed in Section II. In the strict majority rule, the influence from friends may cancel its initial intention to buy the product due to the advertisement. The rating transition diagram is illustrated in Fig. 13.

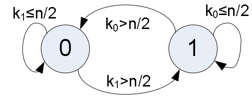


Fig. 13: State transition diagram of strict majority rule.

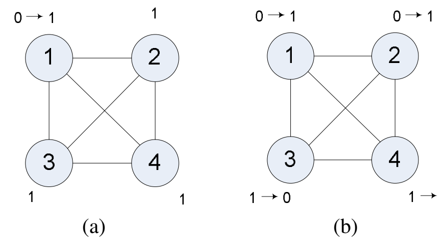


Fig. 14: Example of rating update. (a) Stable case. (b) Unstable case.

Consider a simple social network with four nodes as shown in Fig. 14. Suppose we use the influence model defined by (29) and (30). In Fig. 14(a), there are three users with rating one due to the initial advertisement. Then the network goes to a stable state where all users will give a rating of one. However, when there are exactly two users with rating one initially, the rating of all users will oscillate between zero and one, so the social network is not stable. This is illustrated in Fig. 14(b). In general, instability may occur if there is a loop of state transitions in the mutual influence function, e.g., in the strict majority rule model, the user can change rating from zero to one, and then change back to zero. When social network is not stable, we can not decide the final rating for some users. In the rest of this section, we focus on a special class of influence models that do not allow such a loop of transitions so that we can always decide whether a user will purchase the product or not under these influence models.

We define a class of *monotonous influence models* which have mutual influence function in the following form:

$$f_{s,s+1}(n;\boldsymbol{k}) = 1-f_{s,s}(n;\boldsymbol{k}) \quad (31)$$
$$f_{s,r}(n;\boldsymbol{k}) = 0, \qquad \text{for } r\neq s,s+1, \text{or } k_r=0 \quad (32)$$
$$f_{s,r}(n;\boldsymbol{k}) \leq f_{s,r}(n;\boldsymbol{k}'), \qquad \text{if } \boldsymbol{k}<\boldsymbol{k}'. \quad (33)$$

That is, users' rating never drops and when it increases, it can only increase by one for each update and after the increment, the rating should not exceed the highest rating of its neighbors. Fig. 15 depicts the transition diagram.

We now extend the Algorithm in Section IV to determine the probability that a node will eventually give a rating of s.

Monte Carlo Algorithm

1) for all $0\leq s\leq t$, $c_s\leftarrow$0,
2) repeat for N rounds where N is sufficiently large:

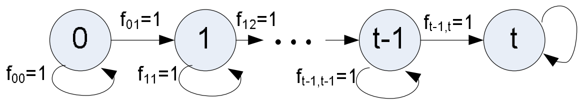


Fig. 15: State transitions for monotonous influence models.

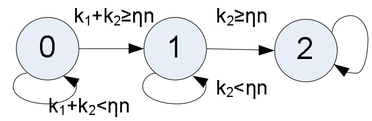


Fig. 16: State transition diagram of the 3-state majority rule.

   a) initialize every node with state 0.
   b) assign every node state (or rating) s with probability p_s.
   c) repeat until no node changes state:
      - for every node, update its state according to the mutual influence function $f_{rs}(n;\boldsymbol{k})$.
   d) if node 1 is at state s, $c_s \leftarrow c_s + 1$.
3) Prob$[X_1 = s] \leftarrow c_s/N$.

One can show that, for the monotonous influence models, the Monte Carlo Algorithm always ends in finite steps. To see this, in each round of step 2c, at least one node increases its rating by one. Since the total rating of all nodes is upper bounded by $t|V|$, the loop in 2c finishes in finite number of rounds. For monotonous influence model on a tree topology, k_i means the number of children who have rating i. Again, we have an equivalent algorithm which uses the shortest path tree T_1 to determine the state of node 1.

Enhanced Algorithm

1) for all $0 \le s \le t$, $c_s \leftarrow 0$,
2) repeat for N rounds where N is sufficiently large:
   a) initialize every node with state 0.
   b) construct the shortest path tree T_1 for node 1, m is the maximal depth.
   c) for $i = m, \ldots, 0$:
      - for every node in generation i, assign it state s with probability p_s. Increase its rating according to the mutual influence function $f_{rs}(n;\boldsymbol{k})$ until it can not increase any more.
   d) If node 1 is in state s, $c_s \leftarrow c_s + 1$.
3) Prob$[X_1 = s] \leftarrow c_s/N$.

To prove the equivalence of the two algorithms above, one can use similar arguments as those in Section IV-A. Suppose we can not ignore the influence from upper generations when updating the state of a node in the enhanced algorithm. There must be a node A with rating l propagating its influence down the tree and raises the rating of a descendant node, and this update triggers a chain of updates up the tree. Since each update can raise the states of nodes to at most l according to (32), it can not change the state of A, thus has no impact on the root. So the influence from a node to its descendants can be ignored when calculating the final state of the root.

Based on this enhanced algorithm, we can calculate the state distribution of a randomly chosen node by the following *iterative distributional equations* (IDE):

$$\text{Prob}[Y_g = s] = \sum_{r=0}^{t} \text{Prob}[\phi_g = r]\text{Prob}[Y_g = s|\phi_g = r] \quad \text{for } g = m, m-1, \ldots, 0. \tag{34}$$

Here random variable Y_g is the state of a node in the g^{th} generation after the update in Step 2c of the Enhanced Algorithm. Random variable ϕ_g is the initial state of a node in the g^{th} generation and $\phi_g \sim \phi$ for all g. Equation (34) computes the distribution of each generation in a bottom-up order. We consider the infinite scale-free graph, so m is infinite and $Y_g \sim Y$ for all $1 \le g \le m$, and (34) reduces to the *recursive distributional equation* (RDE):

$$\text{Prob}[Y = s] = \sum_{r=0}^{t} \text{Prob}[\phi = r]\text{Prob}[Y = s|\phi = r] \tag{35}$$

$$\text{Prob}[X = s] = \sum_{r=0}^{t} \text{Prob}[\phi = r]\text{Prob}[X = s|\phi = r]. \tag{36}$$

Here $X = Y_0$ is the state of the root. The conditional probability Prob$[Y = s|\phi = r]$ is determined by the degree distribution of the underlying random graph and the influence model. Although (35) and (36) are derived for trees, they also apply to random graphs with high accuracy. In the rest of this section, we will illustrate the utility of this methodology by analyzing one particular monotonous influence model.

A. 3-State Majority Rule

Now we consider the *3-state majority rule* model. It has three states $0, 1, 2$ and the acceptance line is $a = 2$. So the fraction of users who buy the product is Prob$[X = 2]$. The mutual influence function is as follows:

$$f_{01}(n;\boldsymbol{k}) = 1 - f_{00}(n;\boldsymbol{k}) = \mathbf{1}[k_1 + k_2 \ge \eta n] \tag{37}$$

$$f_{12}(n;\boldsymbol{k}) = 1 - f_{11}(n;\boldsymbol{k}) = \mathbf{1}[k_2 \ge \eta n] \tag{38}$$

$$f_{22}(n;\boldsymbol{k}) = 1. \tag{39}$$

In other words, a user will increase its rating by one if at least η fraction of its friends give higher rating. The state transition diagram is shown in Fig. 16.

To solve the RDE (35) and (36) for the 3-state majority rule model, we only need to determine Prob$[Y = s|\phi = r]$ and Prob$[X = s|\phi = r]$. We show the derivation of Prob$[Y = s|\phi = r]$, the probability that a non-root user at state r can go to state s. The derivation for other Prob$[X = s|\phi = r]$ is similar.

For 3-state majority rule, by definition, we easily get:

$$\text{Prob}[Y = 0|\phi = 0] = \text{Prob}[k_1 + k_2 < \eta n] \tag{40}$$

$$\text{Prob}[Y = 1|\phi = 0] = \text{Prob}[k_2 < \eta n] - \text{Prob}[k_1 + k_2 < \eta n] \tag{41}$$

$$\text{Prob}[Y = 1|\phi = 1] = \text{Prob}[k_2 < \eta n] \tag{42}$$

$$\text{Prob}[Y=2|\phi=1] = 1-\text{Prob}[k_2<\eta n] \quad (43)$$

$$\text{Prob}[Y=0|\phi=1] = \text{Prob}[Y=0|\phi=2] = \text{Prob}[Y=1|\phi=2]=0 \quad (44)$$

$$\text{Prob}[Y=2|\phi=2] = 1. \quad (45)$$

To derive $\text{Prob}[Y=2|\phi=0]$, note that to go from state 0 to state 2, it has to go through state 1 first. Thus we have:

$$\begin{aligned}\text{Prob}\ [Y=2|\phi=0] &= \text{Prob}[Y_m=1|\phi=0] \\ &\quad \times \text{Prob}[Y=2|Y_m=1,\phi=0] \\ &= \text{Prob}[k_1+k_2\geq \eta n]\times\text{Prob}[k_2\geq\eta n|k_1+k_2\geq\eta n] \\ &= \text{Prob}[k_2\geq\eta n]=1-\text{Prob}[k_2<\eta n]. \quad (46)\end{aligned}$$

Now all we have to do is to derive $\text{Prob}[k_2<\eta n]$ and $\text{Prob}[k_1+k_2<\eta n]$ according to $P_1(k)$, the degree distribution of the descendant nodes of the underlying random graph:

$$\text{Prob}[k_1+k_2<\eta n] = \sum_{k=0}^{\lceil\eta n\rceil-1} P_1(n)C_{n-1}^k \times\text{Prob}[Y=1,2]^k\text{Prob}[Y=0]^{n-1-k} \quad (47)$$

$$\text{Prob}[k_2<\eta n] = \sum_{k=0}^{\lceil\eta n\rceil-1} P_1(n)C_{n-1}^k \times\text{Prob}[Y=2]^k\text{Prob}[Y=0,1]^{n-1-k}. \quad (48)$$

Substituting (40)-(48) into (35) and (36), the resulting equation contains only variables like $\text{Prob}[Y=s]$ and $\text{Prob}[X=s]$, and can be solved by numeric methods.

B. Performance Evaluation

In this subsection, we validate the theoretical results in Section V-A. Fig. 17 shows the simulation results as well as the theoretical prediction by (35) and (36) for the 3-state majority model we discussed in Section V-A. The underlying graph is a randomized power law graph with a minimum degree of two, $\gamma=3$ and the degree distributions are shown in Fig. 3(b). In Fig. 17(a), the initial fraction of state two users p_2 is fixed at 0.1. The horizontal axis is the initial fraction of state one users p_1 and the vertical axis is the probability. We simulate the 3-state majority rule with three different majority factor $\eta=0,1,0.5,0.9$. For each η, there are four curves in the graph showing the simulation and theoretical results of the final fraction of state one users $\text{Prob}[X=1]$ as well as the fraction of the state two users $\text{Prob}[X=2]$. Note if the acceptance line is one, then the fraction of users that buy the product is $\text{Prob}[X=1]+\text{Prob}[X=2]$; if the acceptance line is two, than the fraction of buyers is $\text{Prob}[X=2]$. Fig. 17(b) shows the same curve when p_1 is fixed as 0.1 and p_2 is the horizontal axis. As we see from the figures, the theoretical curves fit the simulation results very well. We observe that when η is small, almost all users give the highest rating. When η is about 0.5, rating one and rating two dominate the users and when the initial fraction of rating two is large enough, rating two finally dominates the whole network. So if $\eta=0.5$ and the acceptance line is two, it is enough to ensure 20% initial buyers in order to enjoy nearly 100% buyers.

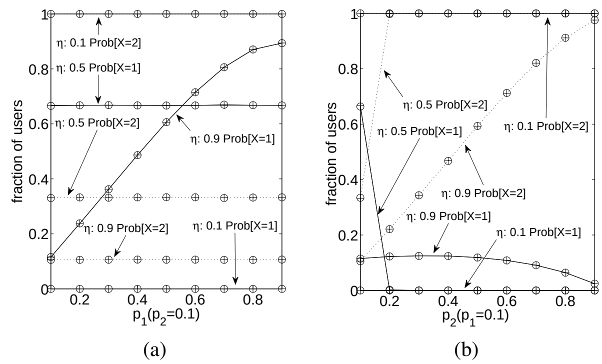


Fig. 17: Impact of the 3-state majority influence model. (a) $p_2=0.1$. (b) $p_1=0.1$.

VI. Related Work

Some of the related work on this problem concern with epidemic spreading [8], [14] via the Susceptible-Infective-Susceptible (SIS) or the Susceptible-Infective-Removed (SIR) model. A body of physical literatures also discussed epidemic spreading on topologies with geographical [15]–[17], community [18], [19], and clustering [20] features for SIS or SIR infection model with one source [2], [21] or Erdös-Rényi graph. Results on the speed the epidemic spreads and dies out are obtained. In this work, we consider more general influence models and focus on the final fraction of users that are influenced. Some researchers discuss specific influence models via algorithmic perspective and design approximation algorithms [22], heuristic algorithms [23] for restricted graphs, or prove NP-hardness results of choosing the most influential nodes [1] in social networks. Some researchers also study the revenue maximization instead of influence maximization, see, [24], [25]. Lastly, previous work of [3], [5] which study security investment game provide local mean field analysis on infinite random graphs. In [26], [27], the authors discussed m-threshold, majority rule and the generalized influence model. In this paper, we consider the spreading of the word-of-mouth effect in viral market, and our objective is to estimate the final fraction of buyers. Moreover, the underlying graph structure we considered is scale free with high clustering coefficient, which has very different properties from random graphs and is also more challenge to analyze.

VII. Conclusion

We first propose a general analytical framework to model various influence mechanisms on scale-free random networks, then extend it for scale-free graphs with high clustering coefficient and influence mechanisms with multiple rating levels. We first discuss the probabilistic model, then we present the deterministic threshold models. Based on these influence models, we compute the expected fraction of users who will eventually purchase the product by applying the local mean field analysis. This metric is important for product advertisement because it reveals the maximum profit that the company can obtain. It also gives us the insights on how to control word-of-mouth effect so as to maximize the revenue. We validate our theoretic analysis by carrying out extensive simulations on scale free graphs with power law degree distribution and high clustering coefficient. We show that our models are very accurate and computational efficient when compared with simulations. We observe that even with a small initial investment of free samples (e.g., small value of ρ), one can still induce large number of users to purchase the product. We also find that clustering coefficient of the underlying network enhances advertising for q-influence and m-threshold model, but impedes advertising for majority rule model. Lastly, our framework provides an important building block to design and analyze different product advertisement strategies in social networks.

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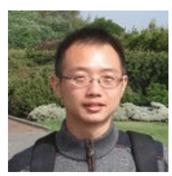

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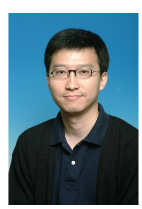

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