Practice Midterm 1

1. Are the propositions "Every two people have a common friend" and "Every person has at least two friends" logically equivalent? Justify your answer.

   Solution: They are not logically equivalent. Suppose the world consists of Alice, Bob, Charlie, and Dave, and the following friendships: Alice with Bob, Bob with Charlie, Charlie with Dave, Dave with Alice. Then every person has two friends, but Alice and Bob have no common friend.

2. Show that for every real number x, at least one of the numbers x, $x + \sqrt{2}$ is irrational.

   Solution: We prove this by contradiction. Assume that there exists a real number x such that both x and $x+\sqrt{2}$ are rational. The difference of two rational numbers is rational, so $(x+\sqrt{2})-x = \sqrt{2}$ is then rational. This contradicts Theorem 9 from Lecture 2.

3. Alice has an infinite supply of \$4 stamps and exactly three \$7 stamps. Can she obtain all integer postage amounts of \$18 and above? Justify your answer.

   Solution: Yes. We prove this by strong induction on the postage amount n. When $n = 18$ (the base case), she can obtain \$18 from two \$7 stamps and one \$4 stamp. Now assume this is true for all postage amounts from \$18 up to \$$n$. She can then make \$$(n+1)$ as follows: If $n+1 = 19$, she uses one \$7 and three \$4 stamps. If $n+1 = 20$ she uses four \$5 stamps. If $n+1 = 21$ she uses three \$7 stamps. If $n+1 \geq 22$, then $n-3 \geq 18$ so by inductive assumption she can make \$$(n-3)$ using \$4 stamps and at most three \$7 stamps. Using an additional \$4 stamp she obtains \$$(n+1)$. It follows that she can obtain any amount above \$18 by strong induction on n.

4. Show that for every integer $n \geq 1$, $1 + 1/4 + 1/9 + \cdots + 1/n^2 \leq 2 - 1/n$.

   Solution: We prove the proposition by induction on n. In the base case $n = 1$, the left hand side is 1 and the right hand side is $2 - 1/1 = 1$, so the proposition holds. Now take any $n \geq 1$ and assume that
   $$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$
   Therefore
   $$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2}.$$
   We can bound the expression on the right like this:
   $$2 - \frac{1}{n} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{n(n+1)} = 2 - \frac{(n+1)-1}{n(n+1)} = 2 - \frac{1}{n+1}$$
   so the inductive conclusion holds. By induction, the predicate is true for all n.

   A side note: How did I come up with the inequality
   $$2 - \frac{1}{n} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{n(n+1)}?$$

I did so by working backwards. In order to complete the inductive step, what we *needed* to prove is that

$$2 - \frac{1}{n} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1}.$$

Moving the terms around, I notice that this is equivalent to proving that

$$\frac{1}{(n+1)^2} \leq \frac{1}{n} - \frac{1}{n+1}.$$

Writing a common denominator for the left hand side, this is the same as

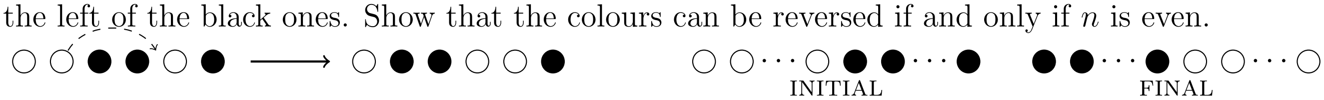
$$\frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)}.$$

This is certainly true, as the denominator on the right is smaller, so the fraction is larger. This reasoning suggests that a good way to proceed is to bound the term $1/(n+1)^2$ by $1/n(n+1)$. After that we just need to simplify the expression.

5. There are 7 girls and 15 boys in a group. Show that some girl is friends with at least five boys or some boy is friends with at most one girl (or both).

   Solution: We prove this by contradiction. Assume that the statement is false, namely every girl has at most four boy-friends and every boy has at least two girl-friends. Let G be a bipartite graph whose vertices are the girls and the boys and whose edges are the girl-boy friendships. Then each girl-vertex has degree at most 4 and each boy-vertex has degree at least 2. On the one hand, the total number of edges equals the sum of the girl-degrees, which is at most $7 \cdot 4 = 28$. On the other hand, this number equals the sum of the boy-degrees, which is at least $15 \cdot 2 = 30$. This is a contradiction.

6. n white pegs and n black pegs are arranged in a line. In each step you are allowed to move any peg past *two* consecutive pegs of the opposite colour, left or right. Initially all white pegs are to the left of the black ones. Show that the colours can be reversed if and only if n is even.



   Solution: First we show that if n is even the colours can be reversed. More generally we show by induction on k that this is true for any number k of white pegs and n black pegs (as long as n is even). When $k = 0$ there are no white pegs so there is nothing to reverse. Now we assume k white pegs and n black pegs can be reversed. Given $k + 1$ white pegs and n black pegs, move the rightmost white peg to the right end by jumping two black pegs at a time and leave it there. By inductive hypothesis the remaining $k + n$ pegs can be reversed, so the whole configuration can be reversed.

   Now we show that if n is odd the colours cannot be reversed. Say a pair of pegs is *inverted* if one is black, one is white, and the black one is to the left of the right one. We prove the following invariant: After any number of steps, the number of inverted pairs is even. This is initially true as the number of inverted pairs is zero. Now assume it is true after t steps. In step $t + 1$, the number of inverted pairs goes up by two if a white peg jumps to the right or a black peg jumps to the left, or down by two if a white peg jumps to the left or a black peg jumps to the right. In all cases, the number of inverted pairs stays even.

   In the final configuration, every one of the n^2 black-white pairs is inverted. Since n is odd, n^2 is also odd so there is an odd number of inverted pairs. Therefore the final configuration can never be reached.

Practice Midterm 2

1. Is the following deduction rule valid?

$$\frac{\forall x \exists y\colon P(x,y) \qquad \exists x \forall y\colon P(x,y)}{\forall x \forall y\colon P(x,y)}$$

Solution: No. Suppose $P(x,y)$ means "person x is happy on day y", Alice is happy on Monday, Alice is happy on Tuesday, Bob is happy on Monday, but Bob is not happy on Tuesday. Then $\forall x \exists y\colon P(x,y)$ is true because everyone is happy sometimes – on Monday, $\exists x \forall y\colon P(x,y)$ is true because someone (Alice) is happy all the time, but $\forall x \forall y\colon P(x,y)$ is false because Bob is unhappy on Tuesday, so not everyone is happy all the time.

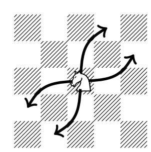
2. Prove that if $m^2 + n^2$ is even then $m + n$ is even.

Solution: We prove the contrapositive: If $m + n$ is odd then $m^2 + n^2$ is odd. Assume $m + n$ is odd. We consider two cases: If m is odd and n is even, then m^2 is odd and n^2 is even, so $m^2 + n^2$ is odd. If m is even and n is odd the same reasoning works with the roles of m and n exchanged. As the two cases cover all possibilities, the statement is true.

3. Show there exists a Die Hard scenario with three jugs and a 1 litre target in which Bruce dies if he can only use any two out of the three jugs to measure, but he survives if he uses all three jugs.

Solution: Suppose Bruce has a 6 litre, a 10 litre, and a 15 litre jug. Since 2 divides both 6 and 10, Bruce cannot measure 1 litre using these two jugs. Since 3 divides both 6 and 15, Bruce also cannot measure 1 litre with them. Since 5 divides both 10 and 15, Bruce dies if he restricts himself to using those two only. However, if all three jugs are available, Bruce can measure 1 litre by filling the 6 and 10 litre jugs to the top, then pouring out their contents into the 15 litre jug until it fills up. There will be 1 litre left in one of them.

4. A knight jumps around an infinite chessboard. Owing to injury it can only make the moves shown in the diagram. Can it ever reach the square immediately to the left of its initial one?



Solution: No. We represent each chessboard square by its integer (x,y) coordinates, with the initial square being $(0,0)$ and the one to its left $(-1,0)$. If we model the jumping knight by a state machine, the transitions out of state (x,y) are

$$(x,y) \to (x-2, y-1) \quad (x,y) \to (x-1, y-2) \quad (x,y) \to (x+2, y+1) \quad (x,y) \to (x+1, y+2).$$

The predicate "3 divides $x + y$" is an invariant of this state machine. It holds initially, and after every transition $(x,y) \to (x',y')$ we have $x'+y' = x+y-3$ in the first two cases and $x'+y' = x+y+3$ in the other two. Assuming the invariant holds before the transition (i.e., 3 divides $x+y$) it also holds after the transition (3 divides $x'+y'$).

The invariant does not hold for state $(-1,0)$ so that state cannot be reached.

5. The vertices of a graph are the integers from 101 to 200 and their cube roots (200 in total). The pair $\{x, y\}$ is an edge if (and only if) $x + y$ is irrational. Does the graph have a perfect matching?

   Solution: No. The graph has 200 vertices, out of which there are at least 101 integers: the numbers 101 to 200 plus $5 = \sqrt[3]{125}$. No pair of integers forms an edge as their sum is rational. If a perfect matching existed, these 101 integers would have to be perfectly matched to the 99 remaining vertices, which is clearly impossible.

   (This is not an instance of Hall's theorem. Hall's theorem is about bipartite graphs and this one is not as $\sqrt[3]{2}$, $\sqrt[3]{3}$ and $\sqrt[4]{4}$ is a cycle of length three.)

6. You are given a graph with 9 men and 9 women as vertices and all possible 81 man-woman pairs as edges. Let Ξ be any matching in this graph. Remove the edges in Ξ (but not the vertices.) Show that the remaining graph has a perfect matching.

   Solution: We show that for every set X of men, $|N(X)| \geq |X|$, where $N(X)$ is the set of neighbours of X. The existence of a perfect matching then follows from Hall's Theorem.

   The proof is by cases. If $1 \leq |X| \leq 8$, take any man x in X. Initially, x had 9 neighbours and at most one of its incident edges was removed, so x has at least 8 neighbours and so $|N(X)| \geq 8$. If $|X| = 9$, then $|N(X)| = 9$ because after the removal of Ξ every woman is still connected to at least one man (in fact, to eight of them).

Practice Midterm 3

1. Underline and explain the mistake in the following "proof."

   Theorem. Every graph has a vertex of even degree.

   Proof. By induction on the number of vertices n. When $n = 1$ the graph has a vertex of degree zero, which is even. Now assume it is true for graphs with n vertices. Let G be a graph with $n + 1$ vertices. Remove any vertex from G. By inductive hypothesis the remaining graph G' has a vertex v of even degree. Since v is also a vertex of G, G has a vertex of even degree. □

   Solution: If v has even degree in G' we cannot conclude that v has even degree in G. The degrees of v in G and G' may be of different parity. For example, if G has two vertices and one edge then v has degree 1 in G but it has degree 0 in G'.

2. Prove that for every integer n there exists an integer k such that $|n^2 - 5k| \leq 1$.

   Solution: First we check that for all n, n^2 mod 5 equals 0, 1 or 4:

n mod 5	0	1	2	3	4
n^2 mod 5	0	1	4	4	1

   Since $4 \equiv -1 \pmod 5$ it follows that for every n, n^2 is congruent to 0, 1, or -1 modulo 5. Therefore n^2 is of the form $5k$ or $5k - 1$ or $5k + 1$ for some integer k. In all cases $|n^2 - 5k| \leq 1$.

3. Alice has infinitely many \$6, \$10, and \$15 stamps. Can she make all integer postages above \$30?

   Solution: Alice can make all integer postages from \$30 to \$35 as follows:

$$\begin{aligned}
\$30 &= 5 \times \$6\\
\$31 &= \$6 + \$10 + \$15\\
\$32 &= 2 \times \$6 + 2 \times \$10\\
\$33 &= 3 \times \$6 + \$15\\
\$34 &= 4 \times \$6 + \$10\\
\$35 &= 2 \times \$10 + \$15
\end{aligned}$$

Now we show that she can make any amount n above 30 by strong induction on n. We already covered the cases $30 \leq n \leq 35$. Now assume that $n > 35$ and she can make all amounts between \$30 and \$$n$. Then $n - 6 \geq 30$ and by inductive assumption she can make $n - 6$ dollars. By adding one \$6 stamp she obtains n dollars.

4. Bob has 32 *b*lue, 33 *r*ed, and 34 *g*reen balls. At every turn he takes out two balls and replaces them with two different balls by the rule below. Can he obtain 99 balls all of the same color?

   replacement rule: $bg \to rr \quad gr \to bb \quad rb \to gg \quad rr \to bg \quad bb \to gr \quad gg \to rb$

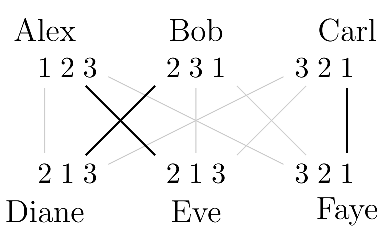
   Solution: We can represent this process by a state machine with states (B, R, G) indicating the number of balls of each color, start state $(32, 33, 34)$, and transitions from (B, R, G) to the states $(B-1, R-1, G+2)$, $(B+2, R-1, G-1)$, $(B-1, R-1, G+2)$, $(B+1, R-2, G+1)$, $(B-2, R+1, G+1)$, $(B+1, R+1, G-2)$ as long as all numbers remain non-negative. The predicate $R - B \equiv 1 \pmod 3$ is an invariant: It holds in the start state and it is preserved by all transitions as $R - B$ can only change by -3, 0, or 3. If all 99 balls are of the same color then $R - B \equiv 0 \pmod 3$, so such a state cannot be reached.

5. Use induction to show that for every $n \geq 1$, the $(n+1) \times n$ grid can be tiled using *two* sets of the following tiles: $1 \times 1, 1 \times 2, \ldots, 1 \times n$.

   Solution: In the base case $n = 1$, we tile the 2×1 grid by putting two 1×1 tiles side by side. For the inductive step, assume that the $(n+1) \times n$ grid (where $n \geq 1$) can be tiled using two sets of the tiles 1×1 up to $1 \times n$. We show that the $(n+2) \times (n+1)$ grid can be tiled using two sets of the tiles 1×1 up to $1 \times (n+1)$: Take the tiling of the $(n+1) \times n$ grid, add an $(n+1) \times 1$ horizontal tile to the top of it and a $1 \times (n+1)$ vertical tile to the right of it. We obtain the desired tiling of the $(n+2) \times (n+1)$ grid.

   By induction, it follows that the proposition is true for all $n \geq 1$.

6. Find a stable matching for these preferences and show that there is no other stable matching.



   Solution: Consider the marked matching {Alex, Eve}, {Bob, Diane}, {Carl, Faye}. We show that no other matching is stable. As a stable matching always exists, this one must be stable.

   In any stable matching, Carl must be matched to Faye because they are each other's first choice (so they would be a rogue couple if not matched). For the rest, the matching {Alex, Diane}, {Bob, Eve} can be ruled out because Bob and Diane would be a rogue couple. This leaves the above matching as the only stable possibility.

Alternative solution: If we run the Gale-Shapley algorithm, on day 1 Alex proposes to Diane and Bob and Carl propose to Faye. Faye picks Carl, so on day 2 both Alex and Bob propose to Diane. Diane picks Bob, so the final matching is $\{\text{Alex}, \text{Eve}\}, \{\text{Bob}, \text{Diane}\}, \{\text{Carl}, \text{Faye}\}$. We proved in Lecture 5 that this is stable.

Let us now run the Gale-Shapley algorithm again, but with the girls doing the proposing this time around. On day 1 Diane and Eve propose to Bob and Faye proposes to Carl. Carl picks Faye and Bob picks Diane over Eve. On day 2 Eve proposes to Alex resulting in the same final stable matching.

By Theorem 6 in lecture 5, the first matching is the best possible for the boys (every boy gets his best possible choice among all stable matchings), while the second one is the worst possible for the boys (every boy gets his worst possible choice). Since they are the same there can be only one stable matching.