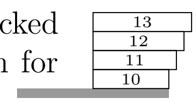
Each question is worth 10 points. Explain your solution clearly and concisely. Write in full sentences. Define all terms and notations that you use but were not introduced in class.

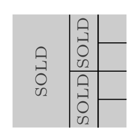
Practice Final 1

1. Write the proposition "There is at most one ball in every urn" using logical connectives and quantifiers. Use the symbols b_1, b_2 for balls, u_1, u_2 for urns and $IN(b, u)$ for "ball b is in urn u".
2. A *cut-edge* in a connected graph is an edge e such that if e was removed, the graph would no longer be connected. Show that any connected graph in which all vertices have even degree does not have a cut-edge.
3. The graph G_1 consists of a single vertex. For $n \geq 1$, the graph G_{n+1} consists of two disjoint copies of G_n and a matching between the vertices of the two copies. How many edges does G_n have?
4. Let $f(n) = 1 + 1/3 + 1/5 + \cdots + 1/(2n - 1)$. Show that f is $\Theta(\log n)$.
5. You drop 30 balls into 7 urns. Some of the balls are red and some are blue. Show that at least three balls of the same colour land in the same urn.
6. You are dealt 5 random cards from a 52-card deck. What is the probability that the largest face value is a 9? (The face values from smallest to largest are 2 3 4 5 6 7 8 9 10 J Q K A.)
7. You have overhang blocks 10, 11, up to n units long, one of each kind. They are stacked over the table from smallest to largest so that their left edges align. (See diagram for $n = 13$). Show that the configuration is not stable when n is sufficiently large.



Practice Final 2

1. Show that for every integer n, if $n^3 + n$ is divisible by 3 then $2n^3 + 1$ is *not* divisible by 3.
2. The vertices of graph G are the integers from 1 to 20. The edges of G are the pairs $\{x, y\}$ such that $\gcd(x, y) > 1$. How many connected components does G have?
3. What is $1 + (1 + 2) + (1 + 2 + 3) + \cdots + (1 + 2 + 3 + \cdots + 1000)$?
4. An $n \times n$ plot of land (n is a power of two) is split in two equal parts by a North-South fence. The Western half is sold and the Eastern half is split in two equal parts by an West-East fence. The same procedure is applied to the remaining $(n/2) \times (n/2)$ plots until 1×1 plots are obtained (see $n = 4$ example). How many units of fence are used?



5. A department has 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?
6. A password is made of the digits `0`, `1`, ..., `9` and the special symbols `*` and `#`. The password must be 4-6 symbols long and contain at least one special symbol. How many passwords are there?
7. Show that every set of 10 integers, each of them between 0 and 25, contains two distinct subsets S, T *of the same size* such that the sum of the numbers in S equals the sum of the numbers in T.

Practice Final 3

1. Show that if x is irrational and y is any real number then at least one of $x+y$ and $x-y$ must be irrational.

2. A box contains 100 black balls and 99 white balls. In each step Alice takes out two balls of the same colour and puts in one ball of the opposite colour. Can Alice ~~empty the box~~ be left with exactly one ball of each colour in the box?

3. In a group of 15 people, is it possible for each person to have exactly 3 friends? (If Alice is a friend of Bob we assume Bob is also a friend of Alice.)

4. Sort these three functions in increasing order of growth: $\sqrt{n} \cdot \log n$, $n/\sqrt{\log n}$, $\sqrt{n \cdot \log n}$. For your sorted list f, g, h show that f is $o(g)$ and g is $o(h)$.

5. The vertices of graph H are the 20 integers from -10 to 10 except 0. The edges of H are the pairs $\{x, y\}$ such that $x = -y$ or $|y - x| = 1$. How many perfect matchings does H have?

6. How many length 5 passwords are there that contain at least one digit ($\texttt{0}, \texttt{1}, \ldots, \texttt{9}$), at least one `*`, and at least one `#`? No other symbols are allowed.

7. Prove that every tree can have at most one perfect matching.

Practice Final 4

1. Let G be a graph with 10 vertices and 9 edges. Is it true that G must be a tree? Justify your answer.

2. Alice places two pebbles at the opposite corners of an 8 by 8 chessboard. At each step, she can

   - put a new pebble in an empty square, if *exactly one* of its neighbors contains a pebble, or
   - remove a pebble from a square, if *at least one* of its neighbors contains a pebble.

   Neighbors are squares that share a common side. Can the board ever have a single pebble on it?

3. The set S_n consists of all length-n strings with symbols $\{\texttt{A}, \texttt{B}, \texttt{C}\}$ in which every `B` is immediately followed by a `C` (e.g., `BCAC` is in S_4 but `ACAB` is not). Find the value of a for which $f(n)$ is $\Theta(a^n)$.

4. What is the largest integer n for which

$$n \leq 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{9999}}?$$

5. At a party, seven people check in their hats. In how many ways can they be returned so that *exactly one* person receives their own hat? Show your calculations.

6. Show that no matter how you place 17 pieces on an 8 by 8 chessboard, at least two pieces must occupy squares that share a common side or a common corner.

7. Suppose that all arrangements of n plus signs and n minus signs in a row are equally likely. Give a formula for the probability that no two minus signs are adjacent to each other. Specify the relevant sample space and event. Show your calculations.