This homework is optional. Questions 1 to 6 are worth 10 points each. Please turn in solutions to *four* questions of your choice (for both ENGG2440A and ESTR2004 students).
Write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions, give credit to your collaborators on your solution sheet, and follow the faculty guidelines and university policies on academic honesty regarding the use of external references.

Questions

1. How many 8×8 chessboard configurations are there with...
   (a) 8 white rooks, and all must be in different rows and columns?
   (b) 4 white and 4 black rooks, and all must be in different rows and columns?
   (c) 8 white and 8 black rooks, and there is exactly one white and one black in every row and every column? (**Hint:** Derangements.)

2. This question concerns 5-card poker hands. Assume that all hands are equally likely. What is the probability that
   (a) there are (at least) two cards with the same face value?
   (b) exactly three of the four suits are represented?
   (c) there is a single card with the largest face value?
   (d) Your opponent has been dealt $\{7\clubsuit, 7\diamondsuit, 7\heartsuit, 7\spadesuit, 5\spadesuit\}$. Your hand is chosen from the remaining cards. What is the probability you beat her? You need a four-of-a-kind with a face value larger than 7 or a straight flush (five consecutive cards of the same suit).

3. In how many ways can you place 10 balls into 3 boxes if
   (a) The balls and the boxes are labeled (i.e., distinguishable)?
   (b) The balls are indistinguishable and the boxes are labeled?
   (c) The balls and the boxes are both indistinguishable?
   (**Hint:** Show a bijection to the set $\{(x, y, z) \colon 3x + 2y + z = 10, x, y, z \geq 0\}$.)
   (d) The balls are labeled but the boxes are indistinguishable? (**Hint:** First calculate the number of configurations with labeled boxes in which no box is empty.)

4. This question concerns sequences of length k whose entries are numbers between 1 and n.
   (a) How many such sequences are there?
   (b) How many such sequences are there in which at least two entries are the same?
   (c) If $n = 2$, how many sequences are there in which both a 1 and a 2 occur at least once?
   (d) **(Extra credit)** If $k = 20$ and $n = 5$, how many sequences are there in which every number occurs at least once?

5. This question concerns systems of sets where every pair is almost disjoint.

   (a) Use induction to prove the following inclusion-exclusion inequality:

   $$|A_1 \cup \cdots \cup A_n| \geq |A_1| + \cdots + |A_n| - |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_{n-1} \cap A_n|.$$

   (b) Suppose $A_1, \ldots, A_n$ are all of the same size s, they are all subsets of $\mathcal{X}$, and each pair intersects in at most one element. Show that $|\mathcal{X}| \geq n \cdot s - \binom{n}{2}$.

   (c) Let $\mathcal{X}$ be the set of all subsets of size 2 of $\{1, \ldots, t\}$ and $A_i = \{S \in \mathcal{X} : i \in S\}$. Show that the inequality in part (b) is an equality for these sets.

6. In this question you will investigate the best possible advantage for the second player in the game of intransitive dice. Given a set D of three dice, Alice chooses a die, Bob chooses on of the remaining two dice, then they each toss their die and the higher number wins. Let $p(D)$ be the probability that Bob wins, assuming they both choose their die optimally.

   (a) Calculate the value of $p(D)$ for the following three dice:

   Die A: $3, 3, 3, 3, 3, 6$ Die B: $2, 2, 2, 5, 5, 5$ Die C: $1, 4, 4, 4, 4, 4$.

   (b) Let $D = \{A, B, C\}$ be any set of three n-sided dice. Consider the experiment in which all three dice are tossed. Describe the sample space and the probabilities of all possible outcomes.

   (c) If the face values of the outcome are (a, b, c), let E_{AB} be the event $a > b$, E_{ABC} be the event $a > b > c$, and so on. Assume the dice have disjoint sets of face values. Show that

   $$\Pr[E_{AB}] + \Pr[E_{BC}] = 1 + \Pr[E_{ABC}] - \Pr[E_{CBA}].$$

   (d) Use part (c) to show that for every D, $p(D) \leq 2/3$.

   (e) **(Extra credit)** Can you find a set of dice (no restriction on the number of faces) for which $p(D)$ is larger than the value you calculated in part (a)?