Questions 1 to 6 are worth 10 points each. If you are in ENGG 2440A, please turn in solutions to *four* questions of your choice. If you are in ESTR 2004, please turn in solutions to *three* questions of your choice and the Mini-project (worth 30 points).

Write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions, give credit to your collaborators on your solution sheet, and follow the faculty guidelines and university policies on academic honesty regarding the use of external references.

Questions

1. Find and explain the mistakes in the following "proofs".

    (a) **Theorem:** In every group of 5 people there is a person who is friends with at least 3 of them.

    Proof. Let a and b denote two of the five people. The proof is by case analysis. We consider two cases:
    - **Case 1:** a is friends with at least 3 other people in the group.
    - **Case 2:** b is friends with at least 3 other people in the group.

    It follows that at least one of a and b is friends with at least 3 other people, so a person that is friends with at least 3 others always exists. □

    (b) **Theorem:** Every group of 8 people includes a group of 4 friends or a group of 4 strangers.

    Proof. Let a denote one of the eight people. The proof is by case analysis. We consider two cases:
    - **Case 1:** a is friends with at least 4 other people in the group.
    - **Case 2:** a is a stranger to at least 4 other people in the group.

    One of these two cases must hold. Let's discuss Case 1. If all the people who are friends with a are strangers among themselves, this is a group of 4 strangers. Otherwise, at least 3 of them are mutual friends, and together with a they form a group of 4 friends. Now let's do Case 2. If all the people who are strangers to a are friends among themselves, this is a group of 4 friends. Otherwise, at least 3 of them are mutual strangers, and together with a they form a group of 4 strangers. □

    (c) **Theorem:** Every even integer greater than two is the sum of two primes.

    Proof. Let's call the integer in question n. We prove this theorem by induction on even integers n. The base case is $n = 4$ as 4 is the first even integer greater than two.
    Base case $n = 4$: $4 = 2 + 2$, so the theorem holds in this case.
    Inductive step: Let n be an even integer greater than two. Assume that n is the sum of two primes p and q. Then $n + 2$ is the sum of the two primes $p + 2$ and q, so the next even integer $n + 2$ is also a sum of two primes. □

    It follows by induction that every even integer greater than two is the sum of two primes.

2. Prove the following theorems.

   (a) For every odd integer n, $3n^2 - 7$ is a multiple of 4.

   (b) For all positive real numbers x, y, if x is irrational, at least one of the numbers $x^2 - y^2$ and $x + y$ is irrational.

   (c) For every real number x such that $0 \le x \le 3$, $x^4 - 2x^3 + x \le 31$.

3. Use induction to prove the following theorems.

   (a) For every positive integer n, $(2n)! \le 4^n \cdot (n!)^2$.

   (b) For every positive integer n, the sum of the first n *even* positive integers is $n(n + 1)$.

   (c) For every odd n, the n by n square grid with a corner square removed can be tiled using rectangular (I-shaped) 2×1 pieces.

4. Which of these propositions are true and which are false? If a proposition is true, prove it. If it is false, prove its negation. (If you claim a number is irrational, provide a proof or give a reference, for example "By Theorem 9 in Lecture 2, $\sqrt{2}$ is irrational.")

   (a) $\sqrt{2} + \sqrt{5}$ is an irrational number.

   (b) For all positive integers n, if $3^n - 1$ is a prime number then n is a prime number.

   (c) For all irrational numbers x, the number $x^2 - \sqrt{2}$ is irrational.

   (d) **(Extra credit)** $\sqrt{2} + \sqrt{3} + \sqrt{5}$ is an irrational number.

5. You start with the three numbers 2, 4, and 6. In each round, you can take any pair of numbers a and b and replace them by $(a + b)/\sqrt{2}$ and $(a - b)/\sqrt{2}$. The third number stays the same. Can you ever end up with the numbers

   (a) $4 + 2\sqrt{2}$, $2\sqrt{2}$, $-4 + 2\sqrt{2}$?

   (b) 3, 4, 5?

   (c) $\sqrt{2}$, $1 + \sqrt{3}$, and $2 + \sqrt{2}$?

6. You have a system of n switches, each of which can be in one of two states: OFF or ON. There are 2^n possible configurations of this system. For example, when $n = 2$ the four possible configurations for the pair of switches are (OFF, OFF), (OFF, ON), (ON, OFF), and (ON, ON).

   Initially, all switches are OFF. In each step, you are allowed to flip exactly one of the switches. Is there a sequence of flips that makes each possible configuration arise exactly once? For example when $n = 2$ this sequence of flips has the desired property (the number on the arrow indicates the switch that is flipped):

   $$(\text{OFF}, \text{OFF}) \xrightarrow{2} (\text{OFF}, \text{ON}) \xrightarrow{1} (\text{ON}, \text{ON}) \xrightarrow{2} (\text{ON}, \text{OFF})$$

   (a) Show a sequence of flips that works when $n = 3$.

   (b) Prove that for every $n \ge 1$, there exists a sequence of flips for n switches that covers every possible configuration exactly once, starting with the all OFF configuration. (**Hint:** Use induction. You may need to strengthen the proposition.)

   (c) Now suppose that you flip not one but two switches at a time. Prove that for every $n \ge 2$, there is no sequence of flips for n switches that covers every possible configuration exactly once, starting with the all OFF configuration. (**Hint:** Use an invariant.)

   (d) **(Extra credit)** If you flip three switches at a time, for which values of n can the task be accomplished?

ESTR 2004 mini-project You are given a $2^n \times 2^n$ grid, where $n \geq 2$, with *four* squares removed. You are interested in tiling the grid with L-shaped tiles, each occupying 3 squares. This grid is special in that its bottom boundary wraps around back to the top, and its right boundary wraps around on a left. (In mathematical jargon we might say the grid is embedded on the surface of a torus.) For example, this is a valid tiling when $n = 2$:
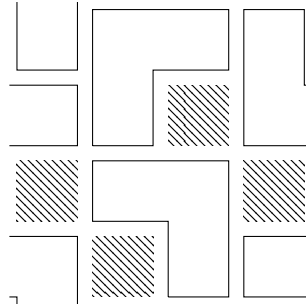
(a) Determine which board configurations admit a tiling that covers every square exactly once. You should specify an easily checkable condition C such that a tiling of the board exists *if and only if* condition C holds. Prove the correctness of your characterization.

(b) Write a computer program that, given as input the number n together with the positions of the four missing squares, outputs a tiling of the board, or prints a suitable message if the board cannot be tiled.

(c) Show the output of your program for the following inputs. The missing squares are specified by the x and y-coordinates of their bottom left corner starting from zero.

n	missing squares
2	$(0,0), (1,1), (2,0), (3,1)$
3	$(0,5), (2,2), (3,1), (6,7)$
3	$(3,3), (4,4), (5,3), (5,5)$
4	$(0,0), (1,1), (13,13), (15,15)$

(d) (**Extra credit**) Repeat parts (a) and (b) for a $3^n \times 3^n$ board with three squares removed.