CSCI5160 Approximation Algorithms — Spring 2020
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Notes 27: John ellipsoid

1. John's theorem

A convex body in $\mathbb{R}^n$ is a closed and bounded convex set with nonempty interior. A convex body K is symmetric if $x \in K$ implies $-x \in K$.

Theorem 1.1. *For any symmetric convex body K in $\mathbb{R}^n$, there is an ellipsoid E such that $E \subseteq K \subseteq \sqrt{n}E$.*

For any (possibly non-symmetric) convex body K in $\mathbb{R}^n$, there is an ellipsoid E such that $E \subseteq K \subseteq n(E-c)+c$, where c is the center of E.

The cube $[-1,1]^n$ shows that the $\sqrt{n}$ factor is tight for symmetric convex body.

2. Convex program

John ellipsoid (or Löwner–John ellipsoid) of a convex body K is the ellipsoid E contained in K of maximum volume.

We will prove John's theorem when K is a symmetric polytope.

To this end, we write a convex program to compute a symmetric ellipsoid E inscribed K of maximum volume.

As in Notes25, a symmetric full-dimensional ellipsoid can be represented as $E = \{y \in \mathbb{R}^n \mid y^\top Q^{-1} y \leqslant 1\}$.

Then $\mathrm{vol}(E) = \sqrt{\det Q} \cdot \mathrm{vol}(\mathbb{B}^n)$, where $\mathbb{B}^n$ is the unit ball in $\mathbb{R}^n$.

Let $K = \{y \in \mathbb{R}^n \mid a_i^\top y \leqslant b_i \text{ for } 1 \leqslant i \leqslant m\}$ be the half-space description of a symmetric polytope K.

That $E \subseteq K$ is equivalent to the supporting hyperplane conditions

$$\text{for } 1 \leqslant i \leqslant m, \qquad \sup\{a_i^\top y \mid y \in E\} \leqslant b_i \qquad \Longleftrightarrow \qquad \sup\{a_i^\top(\sqrt{Q}x) \mid \|x\| \leqslant 1\} \leqslant b_i \,.$$

For supremum in the last inequality, the maximizer is $x_* = \sqrt{Q}a_i/\|\sqrt{Q}a_i\|$, and $a_i^\top(\sqrt{Q}x_*) = a_i^\top Q a_i/\|\sqrt{Q}a_i\| = \|\sqrt{Q}a_i\|$.

So the last inequality is equivalent to

$$\|\sqrt{Q}a_i\|^2 \leqslant b_i^2 \qquad \Longleftrightarrow \qquad a_i^\top Q a_i \leqslant b_i^2 \,.$$

The convex program for John ellipsoid is

$$\begin{aligned} \max\ & \log\det Q \\ & \frac{a_i^\top Q a_i}{b_i^2} \leqslant 1 \qquad \text{for } 1 \leqslant i \leqslant m \\ & Q \succ 0 \end{aligned}$$

This is closely related to the dual convex program in Notes25. Indeed, the two programs are related to each other by duality in projective geometry (planes and points have dual roles).

Maximum inscribed ellipsoid can be found efficiently given half-space representation, just like (dually) minimum circumscribed ellipsoid given vertex representation. By contrast, finding maximum inscribed ellipsoid given vertex representation is NP-hard, and so is (dually) finding minimum circumscribed ellipsoid given half-space representation.

3. Optimality conditions

When the polytope is full dimensional, the convex program is strictly feasiable and Slater's conditions are satisfied. KKT conditions are necessary and sufficient for a solution to be optimal.

Lagrangian of the program is

$$L(Q,\lambda) = -\log\det Q + \sum_{1 \leqslant i \leqslant m} \lambda_i \left(\frac{a_i^\top Q a_i}{b_i^2} - 1 \right) \,.$$

The KKT conditions are

$$\begin{aligned}
\frac{a_i^\top Q a_i}{b_i^2} &\leqslant 1 && \text{for } 1 \leqslant i \leqslant m && \text{Primal feasibility} \\
\lambda_i &\geqslant 0 && \text{for } 1 \leqslant i \leqslant m && \text{Dual feasibility} \\
\lambda_i \left(\frac{a_i^\top Q a_i}{b_i^2} - 1 \right) &= 0 && \text{for } i \leqslant i \leqslant m && \text{Complementary slackness} \\
0 = \nabla_Q L(Q, \lambda) = -(Q^{-1})^\top &+ \sum_{1 \leqslant i \leqslant m} \frac{\lambda_i a_i a_i^\top}{b_i^2} && && \text{Lagrangian optimality}
\end{aligned}$$

In the above, $\nabla \log\det Q = (Q^{-1})^\top$ because

$$\frac{\partial}{\partial Q_{ij}} \log\det Q = \frac{1}{\det Q} \frac{\partial}{\partial Q_{ij}} \det Q = \frac{(-i)^{i+j} M_{ij}}{\det Q} = \frac{\operatorname{adj}(Q)_{ji}}{\det Q}$$

where M_{ij} is the determinant of the $(n-1)$-by-$(n-1)$ matrix by deleting the i-th row and j-th column of Q. Since $Q\operatorname{adj}(Q) = \det(Q)I$, we have $\dfrac{\operatorname{adj}(Q)}{\det Q} = Q^{-1}$ and $\nabla \log\det Q = (Q^{-1})^\top$.

4. Proof

Proof of Theorem 1.1 for symmetric polytope K. Suppose $x \in K$. We want to show $x \in \sqrt{n}E$ (equivalently, $x^\top Q^{-1} x \leqslant n$).

Since K is symmetric, if $a_i^\top x \leqslant b_i$, then $a_i^\top(-x) \leqslant b_i$, and hence $(a_i^\top x)^2 \leqslant b_i^2$. Also, Lagrangian optimality means $Q^{-1} = \sum_{1 \leqslant i \leqslant m} \dfrac{\lambda_i a_i a_i^\top}{b_i^2}$. Therefore

$$x^\top Q^{-1} x = x^\top \left(\sum_{1 \leqslant i \leqslant m} \frac{\lambda_i a_i a_i^\top}{b_i^2} \right) x = \sum_{1 \leqslant i \leqslant m} \frac{\lambda_i (a_i^\top x)^2}{b_i^2} \leqslant \sum_{1 \leqslant i \leqslant m} \lambda_i \,.$$

Finally,

$$n = \operatorname{Tr}(QQ^{-1}) = Q \bullet Q^{-1} = \sum_{1 \leqslant i \leqslant m} \frac{\lambda_i a_i^\top Q a_i}{b_i^2} = \sum_{1 \leqslant i \leqslant m} \lambda_i$$

where the last equality is complementary slackness. □

5. Complementary slackness: geometric interpretation

Complementary slackness conditions have a geometric interpretation.

Apply a linear transformation so that John ellipsoid is the unit ball and $Q = I$ (equivalently, redefine $a_i' = \sqrt{Q} a_i$). Then KKT conditions simplify to

$$\begin{aligned}
\frac{a_i^\top a_i}{b_i^2} &\leqslant 1 && \text{for } 1 \leqslant i \leqslant m && \text{Primal feasibility} \\
\lambda_i &\geqslant 0 && \text{for } 1 \leqslant i \leqslant m && \text{Dual feasibility} \\
\lambda_i \left(\frac{a_i^\top a_i}{b_i^2} - 1 \right) &= 0 && \text{for } i \leqslant i \leq m && \text{Complementary slackness} \\
I &= \sum_{1 \leqslant i \leqslant m} \frac{\lambda_i a_i a_i^\top}{b_i^2} && && \text{Lagrangian optimality}
\end{aligned}$$

Complementary slackness now says that $\lambda_i > 0$ only when $a_i^\top \dfrac{a_i}{b_i} = b_i$. Then $u_i = \dfrac{a_i}{b_i}$ is a contact point of E and K, that is, on both the boundaries of E and K. This is because u_i lies on the hyperplane $a_i^\top x = b_i$, and also $\|u_i\|^2 = u_i^\top u_i = \dfrac{a_i^\top a_i}{b_i^2} = 1$.

Then the dual variables λ_i is supported (only positive) on contact points $\left\{ \frac{a_i}{b_i} \,\middle|\, \lambda_i > 0 \right\}$, and further these contact points $u_i = \frac{a_i}{b_i}$ are in isotropic position when weighted by the dual variables:

$$\sum_{1 \leqslant i \leqslant m} \lambda_i u_i u_i^\top = I \; .$$

The last condition is necessary and sufficient for the unit ball to be the maximum inscribed ellipsoid of a polytope.

6. Remarks

Proof of Theorem 1.1 implies John ellipsoid is unique, because $\log\det$ is strictly concave.

The general case of symmetric convex body that is not a polytope can be proved similarly. The convex program now has infinitely many constraints (a "semi-infinite" program). See [Güler] for more.

The non-symmetric case is also proved by a convex program, where the center of the ellipsoid is part of the variables and the calculations are more complicated.