Collaborating on homework and consulting references is encouraged, but you must write your own solutions in your own words, and list your collaborators and your references. Copying someone else's solution will be considered plagiarism and may result in failing the whole course.

Please answer clearly and concisely. Explain your answers.

(1) Suppose you are given a sequence of nonnegative real numbers $d_{i,j}$ $(1 \leqslant i < j \leqslant n)$, and you want to know whether there are points $v_1, \ldots, v_n$ in the n-dimensional Eucldean space such that their pairwise distances are exactly $d_{i,j}$ (that is, $\|v_i - v_j\|_2 = d_{i,j}$ for all $1 \leqslant i < j \leqslant n$).

Formulate this problem as the feasibility of a semidefinite program.

(2) Let $\mathcal{S} \subseteq \{1, \ldots, n\}$ be a family of subsets over a universe of size n. The following program finds the maximum entropy probability distribution p supported on $\mathcal{S}$ subject to marginal probability constraints:

$$
\begin{aligned}
\max \quad & \sum_{S \in \mathcal{S}} p_S \log \frac{1}{p_S} \\
& \sum_{S \in \mathcal{S}} p_S = 1 \\
\text{for } 1 \leqslant i \leqslant n \quad & \sum_{S \in \mathcal{S},\ S \ni i} p_S = b_i \\
\forall S \in \mathcal{S} \quad & p_S \geqslant 0
\end{aligned}
$$

By considering the optimality condition of the Lagrangian, show that any maximizer satisfying $p_S > 0$ for all $S \in \mathcal{S}$ must be of the form

$$
p_S = \frac{\prod_{i \in S} e^{\lambda_i}}{\sum_{T \in \mathcal{S}} \prod_{i \in T} e^{\lambda_i}}
$$

for some real numbers λ_i.

(3) This problem concerns the Max-Cut problem on a graph G (with positive edge weights), the task of finding $S \subseteq V$ that maximizes $w(S, \overline{S})$ (sum of edge weights across the cut from S to $\overline{S}$). Denote by $\mathrm{MaxCut}(G) = \max_{S \subseteq V} w(S, \overline{S})$.

Below λ_{max} denotes the maximum eigenvalue of a real symmetric matrix.

(a) Show that $\mathrm{MaxCut}(G) \leqslant \frac{n}{4} \lambda_{max}(L)$, with a proof similar to easy side of Cheeger–Alon–Milman.

(b) Show that $\mathrm{MaxCut}(G) \leqslant \min\{\frac{n}{4} \lambda_{max}(L + \mathrm{diag}(u)) \mid u \in \mathbb{R}^n, \sum_{1 \leqslant i \leqslant n} u_i = 0\}$, where $\mathrm{diag}(u)$ denotes the diagonal matrix with u on the diagonal, i.e. $\mathrm{diag}(u)_{ii} = u_i$. Hint: Consider the dual of the SDP used by Goemans–Williamson.

(4) Suppose you are given a function $f : \{0,1\}^n \to \{0,1\}$ that is the OR of some k input bits (and independent of the other input bits), where k is much smaller than n. You do not know which k bits f depends on. You are asked to predict the output of f on a sequence of input strings with the following algorithm: initially set all weights w_i to be 1 for $1 \leqslant i \leqslant n$,

and on input string $x = x_1 \dots x_n$ predict 1 when $w_1 x_1 + \cdots + w_n x_n \geqslant n$, and predict 0 otherwise. You will then learn whether your prediction agrees with $f(x)$. Every time your prediction is wrong, halve or double weights appropriately.

How do you halve or double your weights? Show that you will make $O(k \log n)$ mistakes.

(5) (a) Compute all the eigenvalues of the normalized adjacency matrix of the d-dimensional hypercube graph H_d. Also specify the multiplicities of these eigenvalues.

The hypercube H_d has 2^d vertices that are identified with binary strings of length d. Let $\{0,1\}^d$ denote the set of such strings. Two different vertices $x, y \in \{0,1\}^d$ are adjacent if they agree at $d-1$ positions (and differ at the remaining position).

Hint: First guess a nice eigenbasis for the adjacency matrix.

(b) Show that for every d, the hypercube graph is tight for the easy side of Cheeger–Alon–Milman inequality. (Which subset S has conductance equal to $\lambda_2(\mathcal{L})/2$?)