PDA and CFG conversions

CSCI 3130 Formal Languages and Automata Theory

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CFGs and PDAs

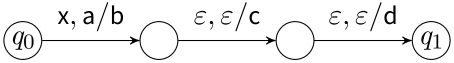
L has a context-free grammar if and only if it is accepted by some pushdown automaton.

$$\begin{tikzcd}\boxed{\text{context-free grammar}} \arrow[r, shift left] & \boxed{\text{pushdown automaton}} \arrow[l, shift left]\end{tikzcd}$$

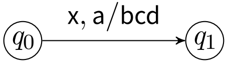
Will first convert CFG to PDA

Convention

A sequence of transitions like



will be abbreviated as



replace a by bcd on stack

Converting a CFG to a PDA

Idea: Use PDA to simulate derivations

$A \rightarrow 0A1$

$A \rightarrow B$

$B \rightarrow \#$

Example:

$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 00B11 \Rightarrow 00\#11$

Rules:

1. Write the start symbol A onto the stack
2. Rewrite variable on top of stack (in reverse) according to production

PDA control		stack	input
write start variable	$\varepsilon, \varepsilon / A$	$\$A$	00#11
replace by production in reverse	$\varepsilon, A / 1A0$	$\$1A0$	00#11

Converting a CFG to a PDA

Idea: Use PDA to simulate derivations

$$A \to 0A1$$
$$A \to B$$
$$B \to \#$$

Example:

$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 00B11 \Rightarrow 00\#11$

Rules:

1. Write the start symbol A onto the stack
2. Rewrite variable on top of stack (in reverse) according to production
3. Pop top terminal if it matches input

PDA control		stack	input
write start variable	$\varepsilon, \varepsilon / A$	$\$A$	00#11
replace by production in reverse	$\varepsilon, A / 1A0$	$\$1A0$	00#11
pop terminal and match	$0, 0 / \varepsilon$	$\$1A$	0#11
replace by production in reverse	$\varepsilon, A / 1A0$	$\$11A0$	0#11

$\vdots$

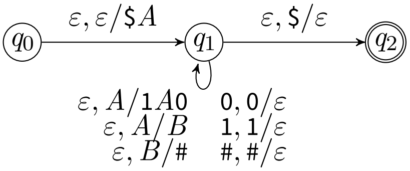
Converting a CFG to a PDA

CFG

$A \rightarrow 0A1$

$A \rightarrow B$

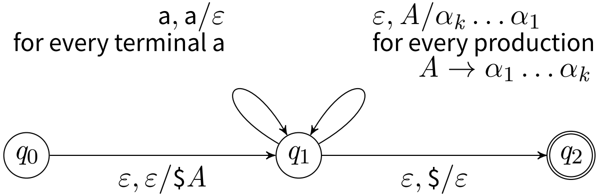
$B \rightarrow \#$



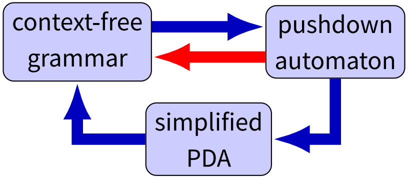
input	00#11	00#11	00#11	00#11	00#11
stack	$\$A$	$\$1A0$	$\$1A$	$\$11A0$	$\$11A$
input	00#11	00#11	00#11	00#11	00#11
stack	$\$11B$	$\$11\#$	$\$11$	$\$1$	$\$$

$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 00B11 \Rightarrow 00\#11$$

General CFG to PDA conversion


a, a/ε
for every terminal a
$\varepsilon, A/\alpha_k \ldots \alpha_1$
for every production
$A \to \alpha_1 \ldots \alpha_k$
q_0
q_1
q_2
$\varepsilon, \varepsilon/\$A$
$\varepsilon, \$/\varepsilon$

From PDAs to CFGs

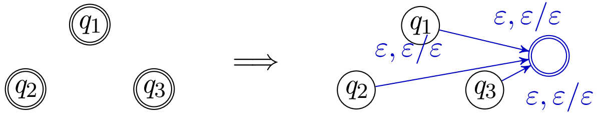


Simplified pushdown automaton:

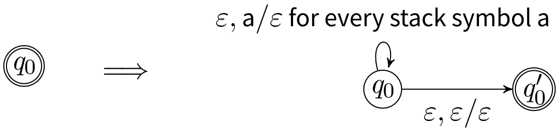
- Has a single accepting state
- Empties its stack before accepting
- Each transition is either a push, or a pop, but not both

Simplifying the PDA

Single accepting state


q_1
q_2
q_3
$\Longrightarrow$
q_1
q_2
q_3
$\varepsilon, \varepsilon/\varepsilon$
$\varepsilon, \varepsilon/\varepsilon$
$\varepsilon, \varepsilon/\varepsilon$

Empties its stack before accepting


q_0
$\Longrightarrow$
ε, a$/\varepsilon$ for every stack symbol a
q_0
q_0'
$\varepsilon, \varepsilon/\varepsilon$

Simplifying the PDA

Each transition either pushes or pops, but not both

$$\begin{tikzcd} q_0 \arrow[r, "{\mathsf{a}, \mathsf{b}/\mathsf{c}}"] & q_1 & \Longrightarrow & q_0 \arrow[r, "{\mathsf{a}, \mathsf{b}/\varepsilon}"] & q'_{01} \arrow[r, "{\varepsilon, \varepsilon/\mathsf{c}}"] & q_1 \end{tikzcd}$$

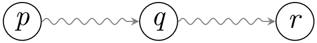
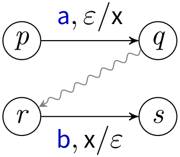
$$\begin{tikzcd} q_0 \arrow[r, "{\mathsf{a}, \varepsilon/\varepsilon}"] & q_1 & \Longrightarrow & q_0 \arrow[r, "{\mathsf{a}, \varepsilon/\mathsf{b}}"] & q'_{01} \arrow[r, "{\varepsilon, \mathsf{b}/\varepsilon}"] & q_1 \end{tikzcd}$$

Simplified PDA to CFG

For every pair (q, r) of states in PDA, introduce variable A_{qr} in CFG

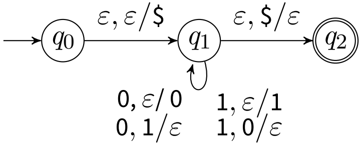
Intention: A_{qr} generates all strings that allow the PDA to go from q to r
(with empty stack both at q and at r)

Simplified PDA to CFG

PDA	CFG
	$A_{qq} \to \varepsilon$
	$A_{pr} \to A_{pq}A_{qr}$
	$A_{ps} \to$ aA_{qr}b a $= \varepsilon$ or b $= \varepsilon$ allowed

Start variable: A_{pq} (initial state p, accepting state q)

Example: Simplified PDA to CFG

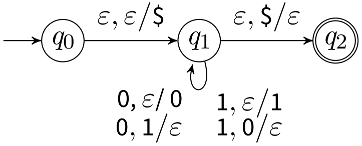


productions:

variables:

start variable:

Example: Simplified PDA to CFG



variables: $A_{00}, A_{11}, A_{22}, A_{01}, A_{02}, A_{12}$

start variable: A_{02}

productions:

$A_{02} \to A_{01}A_{12}$

$A_{01} \to A_{01}A_{11}$

$A_{12} \to A_{11}A_{12}$

$A_{11} \to A_{11}A_{11}$

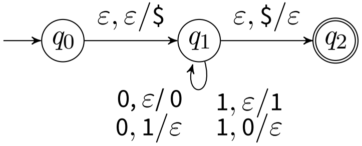
$A_{11} \to 0A_{11}1$

$A_{11} \to 1A_{11}0$

$A_{02} \to A_{11}$

$A_{00} \to \varepsilon,\ A_{11} \to \varepsilon,$
$A_{22} \to \varepsilon$

Example: Simplified PDA to CFG



variables: $A_{00}, A_{11}, A_{22}, A_{01}, A_{02}, A_{12}$

start variable: A_{02}

productions:

$$A_{02} \to A_{01} A_{12}$$

$$A_{01} \to A_{01} A_{11}$$

$$A_{12} \to A_{11} A_{12}$$

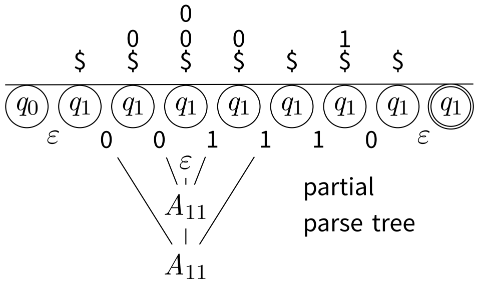
$$A_{11} \to A_{11} A_{11}$$

$$A_{11} \to 0 A_{11} 1$$

$$A_{11} \to 1 A_{11} 0$$

$$A_{02} \to A_{11}$$

$$A_{00} \to \varepsilon,\ A_{11} \to \varepsilon,$$
$$A_{22} \to \varepsilon$$


$
0 0 0 0 1 $ $ $ $ $ $ $
q_0 q_1 q_1 q_1 q_1 q_1 q_1 q_1 q_1
ε 0 0 1 1 1 0 ε
ε
A_{11}
A_{11}

partial parse tree