Notes 03: Separation theorems, Polar sets

1. Convex sets, closed sets

A set $S \subseteq \mathbb{R}^n$ is convex if for every $x, y \in S$ and any $0 \leqslant \lambda \leqslant 1$, the convex combination $\lambda x + (1-\lambda)y \in S$. That means every line segment connecting any two points in S lies in S.

$\langle x, y\rangle = \sum_{1 \le i \leqslant n} x_i y_i$ denotes the inner product in $\mathbb{R}^n$.

Define $\|x\| = \sqrt{\langle x, x\rangle}$ is the Euclidean norm of $x \in \mathbb{R}^n$.

A point $y \in \mathbb{R}^n$ is a limit point of $S \subseteq \mathbb{R}^n$ if for every $\varepsilon > 0$, some $x \in S$ satisfies $\|x - y\| \leqslant \varepsilon$. In other words, y is a limit point if it is arbitrarily close to (some point in) S. A set $S \subseteq \mathbb{R}^n$ is closed if S contains all its limit points.

For example, the open unit disk $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ is not closed, because $(1, 0)$ is its limit point but does not belong to $\mathbb{D}$. By contrast, the closed unit ball $\overline{\mathbb{D}} = \{(x, y) \in \mathbb{R}^2 \mid x^2+y^2 \leqslant 1\}$ (with non-strict inequality) is closed.

2. Separation theorems

Separation theorems are closely related to duality. In fact, some of them are equivalent to strong duality of convex programs. We now state one version, saying a point outside a closed convex set must be separated from the set by a hyperplane.

Theorem 2.1 (Separation theorem)**.** *Let $S \subseteq \mathbb{R}^n$ be a non-empty closed convex set and $v \notin S$. Then there is $y \in \mathbb{R}^n$ such that $\langle y, v\rangle > \langle y, x\rangle$ for all $x \in S$.*

Idea: Given v, find the unique point $x^* \in S$ closest to v. Then argue $\langle v - x^*, x - x^*\rangle \leqslant 0$ for all $x \in S$. This gives a separating hyperplane normal to the direction $v - x^*$.

Claim 2.2. *There exists a unique point $x^* \in S$ closest to v.*

Proof. **(Existence)** Let x be an arbitrary point in S. Let $Z = \{z \in S \mid \|z - x\|^2 \leqslant \|x - v\|^2\}$ be the set of points in S that are as close to v as x. Z is bounded and closed. A point closest to v minimizes the continuous function $f(z) = \|z - v\|^2$ (squared distance between z and x) over Z. Weierstrass' (extreme value) theorem tells us the minimum of f is attained at some point $x^* \in S$.

(Uniqueness) Consider any two minimizers x_1, x_2 of $f(z)$ in Z. We will show that they must in fact be the same. Indeed, consider their midpoint $\overline{x} = (x_1 + x_2)/2$. $\overline{x} \in S$ since S is convex. Let $\mu = \|x_1 - v\|^2 = \|x_2 - v\|^2$ be the minimum squared distance. Then $\|\overline{x} - v\|^2 = \frac{\|x_1 - v\|^2}{2} + \frac{\|x_2 - v\|^2}{2} - \frac{\|x_1 - x_2\|^2}{4} = \mu - \frac{\|x_1 - x_2\|^2}{4}$ $(\leqslant \mu = \|x_1 - v\|^2)$. Since x_1 is a closest point to v, the last inequality must in fact be an equality, hence $\|x_1 - x_2\|^2 = 0$ and $x_1 = x_2$. □

Lemma 2.3. *x^* minimizes $f(z)$ over S if and only if $\langle v - x^*, x - x^*\rangle \leqslant 0$ for all $x \in S$*

Proof. Consider any $x \in S$. Let $z = (1-\varepsilon)x^* + \varepsilon x$ be a point very close to x^* on the line segment between X^* and x. By convexity $z \in S$. Expand $f(z)$ as

$$\|z - v\|^2 = \|(x^* - v) - \varepsilon(x^* - x)\|^2 = \|x^* - v\| - 2\varepsilon\langle x^* - v, x^* - x\rangle + \varepsilon^2\|x^* - x\|^2. \quad (1)$$

The derivative wrt ε at $\varepsilon = 0$ is $-2\langle x^* - v, x^* - x\rangle$, which must be nonnegative for x^* to be a minimizer. Conversely, if $\langle v - x^*, x - x^*\rangle \leqslant 0$, then the last two terms on the right-hand-side of Eq. (1) are nonnegative when $\varepsilon = 1$, so $\|x - v\|^2 = \|z - v\|^2 \geqslant \|x^* - v\|^2$ (because $z = x$ when $\varepsilon = 1$). □

Proof of Theorem 2.1. Let $y = v - x^*$.

$\|v - x^*\|^2 > 0 \iff \langle v - x^*, v\rangle > \langle v - x^*, x^*\rangle \iff \langle y, v\rangle > \langle y, x^*\rangle$

For any $x \in S$, Lemma says $\langle v - x^*, x - x^*\rangle \leqslant 0 \iff \langle v - x^*, x\rangle \leqslant \langle v - x^*, x^*\rangle \iff \langle y, x\rangle \leqslant \langle y, x^*\rangle$

Therefore $\langle y, v\rangle > \langle y, x^*\rangle \geqslant \langle y, x\rangle$ for any $x \in S$. □

There are other versions of separation theorems in [BV §2.5]

3. Supporting hyperplane

Given a set $S \subseteq \mathbb{R}^n$ and a point x_0 on the boundary of S, a hyperplane $\{x \in \mathbb{R}^n \mid \langle y, x\rangle = \langle y, x_0\rangle\}$ is a supporting hyperplane to S at x_0 if $\langle y, x\rangle \leqslant \langle y, x_0\rangle$ for all $x \in S$.

The supporting hyperplane theorem states that for any convex set S and any boundary point x_0, there exists a supporting hyperplane to S at x_0.

It follows from the separation theorem, see [BV §2.5.2].

4. Polar sets

Support functions are an alternative representation of a convex set C.

Definition 4.1. The *support function* of $C \subseteq \mathbb{R}^n$ is $S_C(y) = \sup\{\langle y, x\rangle \mid x \in C\}$.

It tells us how far the set C goes along vector y.

Using separation theorem, one can show that two closed convex sets are equal if and only if they have the same support functions [BV, Exercise 2.26].

All information about a convex set is described by its boundary!

We now define a dual object of a set, called *polar*.

Given a set $C \subseteq \mathbb{R}^n$, the polar of C is $C^\circ = \{y \in \mathbb{R}^n \mid \langle y, x\rangle \leq 1 \quad \forall x \in C\}$.

It is easy to check that C° is a closed convex set, regardless of whether C is convex.

Under an additional assumption, the polar contains all the information about the support function:

Suppose for every $y \in \mathbb{R}^n$, $S_C(y) = \sup_{x\in C}\langle y, x\rangle \geqslant 0$.

Then given $y \in \mathbb{R}^n$, the polar C° encodes the support function of C because

$\mu \geqslant S_C(y) = \sup\{\langle y, x\rangle \mid x \in C\} \iff \langle y, x\rangle \leqslant \mu \quad \forall x \in C \iff \langle \frac{y}{\mu}, x\rangle \leqslant 1 \quad \forall x \in C$ (using $\mu \geqslant 0$) $\iff \frac{y}{\mu} \in C^\circ$

If C is a closed convex set, then we can recover C from C°.

When is the support function nonnegative?

By Separation theorem, the support function is nonnegative if and only if C contains the origin.

We can imagine the polar as (the convex hull of) the set of supporting hyperplanes of C.

For example, the polar of the polyhedron $\{x \in \mathbb{R}^n \mid \langle a_i, x\rangle \leqslant b_i, 1 \leqslant i \leqslant m\}$ is the convex hull of $\{\frac{a_1}{b_1}, \dots, \frac{a_m}{b_m}\}$.

In fact, C can be recovered from C° as the polar of C° (also called the *bipolar of* C).

Theorem 4.2. *If C is a closed convex set that contains the origin, then $C^{\circ\circ} = C$.*

Proof. $C \subseteq C^{\circ\circ}$: $\quad x \in C \implies \forall y \in C^\circ \quad \langle y, x\rangle \leqslant 1$ (by definition of C°) $\iff x \in (C^\circ)^\circ$ (by definition of $C^{\circ\circ}$)

$C^{\circ\circ} \subseteq C$: We will show that if $v \notin C$, then $v \notin C^{\circ\circ}$. To show $v \notin C^{\circ\circ}$, we need to find $y' \in C^\circ$ such that $\langle y', v\rangle > 1$.

If $v \notin C$, then Separation theorem gives us $y \in \mathbb{R}^n$ such that $\langle y, x\rangle < \langle y, v\rangle$ for all $x \in C$. Since C contains the origin, $\langle y, v\rangle > \langle y, 0\rangle = 0$. Let b satisfy $0 \leqslant \sup\{\langle y, x\rangle \mid x \in C\} < b < \langle y, v\rangle$ and $y' = y/b$. Then $\langle y', x\rangle < 1$ for all $x \in C$, hence $y' \in C^\circ$. Also $\langle y', v\rangle > 1$. Therefore $v \notin C^{\circ\circ}$. □