
Collaborating on homework and consulting references is encouraged, but you must write your own solutions in your own words, and list your collaborators and your references. Copying someone else's solution will be considered plagiarism and may result in failing the whole course.

Please answer clearly and concisely. Explain your answers.

(1) (20 points) Suppose you are given a function $f : \{0,1\}^n \to \{0,1\}$ that is the OR of some k input bits (and independent of the other input bits), where k is much smaller than n. You do not know which k bits f depends on.

You are asked to predict the output of f on a sequence of input strings with the following algorithm: initially set all weights w_i to be 1 for $1 \leqslant i \leqslant n$, and on input string $x = x_1 \dots x_n$ predict 1 when $w_1 x_1 + \cdots + w_n x_n \geqslant n$, and predict 0 otherwise. You will then learn whether your prediction agrees with $f(x)$. Every time your prediction is wrong, halve or double weights appropriately.

How do you halve or double your weights? Show that you will make $O(k \log n)$ mistakes.

(2) (15 points) Let G be a graph with positive edge weights and no isolated vertices. Show that G's normalized adjacency matrix has minimum eigenvalue -1 if and only if G has a connected component that is bipartite.

(3) This problem concerns the Max-Cut problem on a graph G (with positive edge weights), the task of finding $S \subseteq V$ that maximizes $w(S, \overline{S})$ (sum of edge weights across the cut from S to $\overline{S}$). Denote by $\mathrm{MaxCut}(G) = \max_{S \subseteq V} w(S, \overline{S})$.

Below $\lambda_{\max}$ denotes the maximum eigenvalue of a real symmetric matrix.

(a) (10 points) Show that $\mathrm{MaxCut}(G) \leqslant \frac{n}{4} \lambda_{\max}(L)$, with a proof similar to easy side of Cheeger–Alon–Milman.

(b) (10 points) Show that

$$\mathrm{MaxCut}(G) \leqslant \min \left\{ \frac{n}{4} \lambda_{\max}(L + \mathrm{Diag}(u)) \;\middle|\; u \in \mathbb{R}^n, \sum_{1 \leqslant i \leqslant n} u_i = 0 \right\},$$

where $\mathrm{Diag}(u)$ denotes the diagonal matrix with u on the diagonal, i.e. $\mathrm{Diag}(u)_{ii} = u_i$. *Hint: Consider the dual of the SDP used by Goemans–Williamson.*

(4) (a) (10 points) Compute all the eigenvalues of the normalized adjacency matrix of the d-dimensional hypercube graph H_d. Also specify the multiplicities of these eigenvalues.

The hypercube H_d has 2^d vertices that are identified with binary strings of length d. Let $\{0,1\}^d$ denote the set of such strings. Two different vertices $x, y \in \{0,1\}^d$ are adjacent if they agree at $d-1$ positions (and differ at the remaining position).

Hint: First guess a nice eigenbasis for the adjacency matrix.

(b) (10 points) Show that for every d, the hypercube graph is tight for the easy side of Cheeger–Alon–Milman inequality, i.e. $\frac{\lambda_2(\mathcal{L})}{2} = \varphi(H_d)$. (Which subset S has conductance equal to $\lambda_2(\mathcal{L})/2$?)

(c) (10 points) Show that the hard side of Cheeger–Alon–Milman inequality is tight up to constant factors by considering the n-cycle graph C_n.

More precisely, show that there is a constant B, independent of n, so that for any n-cycle graph C_n (with $n \geqslant 3$),

$$\sqrt{\lambda_2(\mathcal{L})} \leqslant B\varphi(C_n) \ .$$

Hint: See Section 6.5 of the Spielman reference book.

(5) (15 points) Let G be an undirected graph with positive edge weights. Suppose the degree of every vertex is 1.

Fix a positive parameter t. Given any vertex i, consider the following random process:

(i) Pick a nonnegative integer k from the Poisson distribution with parameter t

(ii) Run the usual random walk for k steps starting from the initial vertex i

Denote by P_{ij} the probability that the above random process ends up at vertex j after the k-step walk, given the initial vertex i. Let P be the matrix with entries P_{ij}.

Compute P and show that $P = \exp(-tL)$, where L is the Laplacian matrix of G. Recall that $\exp X = \sum_{k=0}^{\infty} \frac{X^k}{k!}$ for a real symmetric matrix X.