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A Barrier Function Method for the Nonconvex Quadratic Programming Problem with Box Constraints

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Abstract. In this paper a barrier function method is proposed for approximating a solution of the nonconvex quadratic programming problem with box constraints. The method attempts to produce a solution of good quality by following a path as the barrier parameter decreases from a sufficiently large positive number. For a given value of the barrier parameter, the method searches for a minimum point of the barrier function in a descent direction, which has a desired property that the box constraints are always satisfied automatically if the step length is a number between zero and one. When all the diagonal entries of the objective function are negative, the method converges to at least a local minimum point of the problem if it yields a local minimum point of the barrier function for a sequence of decreasing values of the barrier parameter with zero limit. Numerical results show that the method always generates a global or near global minimum point as the barrier parameter decreases at a sufficiently slow pace.

Key words: Nonconvex Quadratic Programming, Box Constraints, Global Optimization, Barrier Function, Descent Direction, Iterative Method

1. Introduction

The nonconvex quadratic programming problem with box constraints is to minimize a nonconvex quadratic function subject to box constraints. It is an NP-hard problem (Murty and Kabadi, 1987) and has many diverse applications (Pardalos and Rosen, 1987). A special case of the problem is the quadratic zero-one programming problem. In order for a solution of the quadratic zero-one programming problem, many exact algorithms have been developed, such as ones given in Barahona et al. (1989), Carter (1984), Gulati et al. (1984), Hammer and Simeone (1987), Hansen (1979), Pardalos (1991), Pardalos and Jha (1992), Pardalos and Rogers (1990), etc. Most of these algorithms are of the branch-and-bound type or use some type of linearization techniques. In addition a differentiable exact penalty function for the general quadratic programming problem can be found in Grippo and Lucidi (1991). We refer to Floudas and Visweswaran (1995) for an excellent survey of quadratic optimization.

Due to its computational complexity, the nonconvex quadratic programming problem with box constraints is, in general, difficult to solve to optimality. Several approximation algorithms have been proposed, such as ones given in Poljak and Wolkowicz (1995), Vavasis (1992), Ye (1991), etc. A recent survey of algorithms for the nonconvex quadratic programming problem with box constraints can be found in De Angles et al. (1997). For some NP-hard combinatorial optimization problems, numerical results show that deterministic annealing seems effective (Yuille and Kosowsky, 1994). The approach is a heuristic continuation method, which attempts to find the global minimum of the effective energy at high temperature and track it as the temperature decreases. There is no guarantee that the minimum at high temperature can always be tracked to the minimum at low temperature, but the experimental results are encouraging (Durbin and Willshaw, 1987; Peterson, 1990). The nonconvex quadratic programming problem with box constraints is an NP-hard problem. The deterministic annealing approach may provide an alternative solution procedure for the problem.

In this paper we adapt the idea of deterministic annealing for approximating a solution of the nonconvex quadratic programming problem with box constraints. A barrier function method is proposed, which attempts to produce a solution of good quality by following a path as the barrier parameter decreases from a sufficiently large positive number satisfying that the barrier function is strictly convex. For a given value of the barrier parameter, the method searches for a minimum point of the barrier function in a descent direction, which has a desired property that the box constraints are always satisfied automatically if the step length is a number between zero and one. When all the diagonal entries of the objective function are negative, the method converges to at least a local minimum point of the problem if it yields a local minimum point of the barrier function for a sequence of decreasing values of the barrier parameter with zero limit. Numerical results show that the method always generates a global or near global minimum point as the barrier parameter decreases at a sufficiently slow pace.

The rest of this paper is organized as follows. We describe the barrier function and derive some properties in Section 2. We introduce the method in Section 3. We present some numerical results in Section 4 to show that the method is effective and efficient. We conclude the paper with some remarks in Section 5.

2. Barrier Function

The problem we intend to solve is as follows: Find a minimum point of

$$\begin{aligned} &\min \;\; f(x) = \tfrac{1}{2}x^\top Qx + c^\top x \\ &\text{subject to } \; l_i \leqslant x_i \leqslant u_i, \;\; i = 1, 2, \cdots, n, \end{aligned} \tag{2.1}$$

where

$$Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}$$

is symmetric and indefinite or negative semidefinite, and $l_i < u_i$, $i = 1, 2, \cdots, n$. Let $l = (l_1, \cdots, l_n)^\top$, $u = (u_1, \cdots, u_n)^\top$, and

$$B = \{x \mid l \leqslant x \leqslant u\}.$$

We assume that l_i and u_i, $i = 1, 2, \cdots, n$, are finite. Then B is bounded.

To approximate a solution of (2.1), we introduce a barrier term,

$$(x_i - l_i)\ln(x_i - l_i) + (u_i - x_i)\ln(u_i - x_i), \tag{2.2}$$

to incorporate $l_i \leqslant x_i \leqslant u_i$ into the objective function, and obtain a barrier function,

$$e(x, \beta) = f(x) + \beta \sum_{i=1}^{n} ((x_i - l_i)\ln(x_i - l_i) + (u_i - x_i)\ln(u_i - x_i)), \tag{2.3}$$

where β is the barrier parameter that behaves as temperature in the deterministic annealing approach and varies from a positive number to zero. The initial value of β should be sufficiently large so that $e(x, \beta)$ is strictly convex over $l \leqslant x \leqslant u$. Observe that the barrier term (2.2) comes from the entropy function (Fang, Rajasekera and Tsao, 1997), and is well defined at $x_i = l_i$ and $x_i = u_i$ since

$$\lim_{x_i \to l_i^+} (x_i - l_i)\ln(x_i - l_i) = \lim_{x_i \to u_i^-} (u_i - x_i)\ln(u_i - x_i) = 0.$$

When $l_i = 0$ and $u_i = 1$, the barrier term (2.2) appears implicitly as a term of the energy function defined in Hopfield (1984).

Instead of solving (2.1) directly, let us consider a scheme, which obtains a solution of (2.1) from the solution of

$$\min_{x \in B} e(x, \beta)$$

at the limit of $\beta \downarrow 0$. From $e(x, \beta)$, we obtain

$$\frac{\partial e(x, \beta)}{\partial x_i} = \frac{\partial f(x)}{\partial x_i} + \beta \ln \frac{x_i - l_i}{u_i - x_i}.$$

Observe that

$$\lim_{x_i \to l_i^+} \frac{\partial e(x, \beta)}{\partial x_i} = -\infty \text{ and } \lim_{x_i \to u_i^-} \frac{\partial e(x, \beta)}{\partial x_i} = +\infty.$$

The Hessian matrix of $e(x, \beta)$ at x with $l < x < u$ is given by

$$\nabla^2 e(x, \beta) = Q + \beta((X - L)^{-1} + (U - X)^{-1}),$$

where X is the diagonal matrix formed by the components of x, L the diagonal matrix formed by the components of l, and U the diagonal matrix formed by the components of u. When β is sufficiently large, clearly, $\nabla^2 e(x, \beta)$ is positive definite for any x with $l < x < u$. Therefore, $e(x, \beta)$ is strictly convex over $l \leqslant x \leqslant u$ when β is sufficiently large.

Since $\nabla f(x) = Qx + c$ is bounded on B, one can readily derive the following result.

LEMMA 1. *For any given $\beta > 0$, if x^* is a minimum point of*

$$\min_{l \leqslant x \leqslant u} e(x, \beta)$$

then

$$l < x^* < u.$$

Proof. Suppose that some component of x^*, say x_i^*, equals l_i. Let ϵ be a positive number arbitrarily close to zero. We define $y^* = (y_1^*, \cdots, y_n^*)^\top$ by

$$y_j^* = \begin{cases} x_j^* & \text{if } j \neq i, \\ x_i^* + \epsilon & \text{if } j = i, \end{cases}$$

$j = 1, \cdots, n$. Then, when ϵ is sufficiently small,

$$\begin{aligned} \frac{\partial e(y^*, \beta)}{\partial x_i} &= \frac{\partial f(y^*)}{\partial x_i} + \beta \ln \frac{x_i^* + \epsilon - l_i}{u_i - x_i^* - \epsilon} \\ &= \frac{\partial f(y^*)}{\partial x_i} + \beta \ln \frac{\epsilon}{u_i - l_i - \epsilon} < 0 \end{aligned}$$

since $\frac{\partial f(y^*)}{\partial x_i}$ is bounded. Thus, adding to the ith component of y^* an arbitrarily small positive number, one can obtain a point of B arbitrarily close to y^*, at which $e(x, \beta)$ is less than $e(y^*, \beta)$. Because $e(x, \beta)$ is continuous on B, $e(y^*, \beta)$ is arbitrarily close to $e(x^*, \beta)$ if ϵ is arbitrarily close to zero. Therefore, there exists a point of B arbitrarily close to x^*, at which $e(x, \beta)$ is less than $e(x^*, \beta)$. This contradicts that x^* is a minimum point, which implies that no component of x^* equals lower bound. Similarly, one can show that no component of x^* equals upper bound. The lemma follows. □

This lemma indicates that if x^* is a minimum point of

$$\min_{l \leqslant x \leqslant u} e(x, \beta)$$

then

$$\nabla_x e(x^*, \beta) = 0,$$

where $\nabla_x e(x,\beta) = (\frac{\partial e(x,\beta)}{\partial x_1}, \cdots, \frac{\partial e(x,\beta)}{\partial x_n})^\top$.

Let β_k, $k = 1, 2, \cdots$, be a sequence of positive numbers such that

$$\beta_1 > \beta_2 > \cdots$$

and $\lim_{k\to\infty} \beta_k = 0$. Let x^* be a global minimum point of (2.1) and

$$x(\beta_k) = \operatorname{argmin}\{e(x, \beta_k) \mid x \in B\},$$

$k = 1, 2, \cdots$.

THEOREM 1. *For* $k = 1, 2, \cdots$,

$$f(x(\beta_k)) \geqslant f(x(\beta_{k+1})),$$

and

$$\lim_{k\to\infty} f(x(\beta_k)) = f(x^*).$$

Proof. Let

$$p(x) = \sum_{i=1}^{n} ((x_i - l_i)\ln(x_i - l_i) + (u_i - x_i)\ln(u_i - x_i)).$$

Then, for any $x \in B$,

$$\sum_{i=1}^{n} (u_i - l_i)\ln\frac{u_i - l_i}{2} \leqslant p(x) \leqslant \sum_{i=1}^{n} (u_i - l_i)\ln(u_i - l_i).$$

Let

$$b(x) = p(x) - \sum_{i=1}^{n} (u_i - l_i)\ln\frac{u_i - l_i}{2}.$$

Then, $b(x) \geq 0$ for any $x \in B$. Let

$$\psi(x, \beta) = f(x) + \beta b(x).$$

Then,

$$e(x, \beta) = \psi(x, \beta) + \beta \sum_{i=1}^{n} (u_i - l_i)\ln\frac{u_i - l_i}{2}.$$

Thus,

$$x(\beta_k) = \operatorname{argmin}\{\psi(x, \beta_k) \mid x \in B\}.$$

By the definitions of $x(\beta_k)$ and $x(\beta_{k+1})$, we have

$$f(x(\beta_k)) + \beta_k b(x(\beta_k)) \leqslant f(x(\beta_{k+1})) + \beta_k b(x(\beta_{k+1}))$$

and

$$f(x(\beta_{k+1})) + \beta_{k+1}b(x(\beta_{k+1})) \leqslant f(x(\beta_k)) + \beta_{k+1}b(x(\beta_k)).$$

Then,

$$(\beta_k - \beta_{k+1})b(x(\beta_k)) \leqslant (\beta_k - \beta_{k+1})b(x(\beta_{k+1})).$$

Thus,

$$b(x(\beta_k)) \leqslant b(x(\beta_{k+1}))$$

since $\beta_k > \beta_{k+1}$. Therefore,

$$f(x(\beta_k)) \geq f(x(\beta_{k+1})).$$

For any k, we can write

$$f(x^*) \leqslant f(x(\beta_k)) \leqslant f(x(\beta_k)) + \beta_k b(x(\beta_k)) = \psi(x(\beta_k), \beta_k). \tag{2.4}$$

Note that for any $\epsilon > 0$, there exists $\bar{x} \in B$ such that

$$f(\bar{x}) \leqslant f(x^*) + \epsilon.$$

It follows that for any k,

$$f(x^*) + \epsilon + \beta_k b(\bar{x}) \geq f(\bar{x}) + \beta_k b(\bar{x}) \geq \psi(x(\beta_k), \beta_k).$$

Then,

$$\lim_{k\to\infty} \psi(x(\beta_k), \beta_k) \leqslant f(x^*) + \epsilon.$$

From (2.4), we obtain

$$\lim_{k\to\infty} \psi(x(\beta_k), \beta_k) \geq f(x^*).$$

Thus,

$$\lim_{k\to\infty} \psi(x(\beta_k), \beta_k) = f(x^*).$$

Observe that $\lim_{k\to\infty} \beta_k b(x(\beta_k)) = 0$. Therefore,

$$\lim_{k\to\infty} f(x(\beta_k)) = f(x^*).$$

This completes the proof of the theorem. □

This theorem indicates that every limit point of $x(\beta_k)$, $k = 1, 2, \cdots$, is a global minimum point of (2.1).

THEOREM 2. *When every diagonal entry of Q is negative, (2.1) has a vertex global minimum point.*

Proof. Let $x(\beta_{k_j})$, $j = 1, 2, \cdots$, be a convergent subsequence of $x(\beta_k)$, $k = 1, 2, \cdots$. Assume that $\lim_{j\to\infty} x(\beta_{k_j}) = v^*$. From Theorem 1, we obtain $f(v^*) = f(x^*)$. In the following we show that v^* is a vertex of B. Since $x(\beta_{k_j})$ is a minimum point of $\min_{x\in B} e(x, \beta_{k_j})$, hence, the Hessian matrix of $e(x, \beta_{k_j})$ at $x(\beta_{k_j})$,

$$Q + \beta_{k_j}((X(\beta_{k_j}) - L)^{-1} + (U - X(\beta_{k_j}))^{-1}),$$

is positive semidefinite, where $X(\beta_{k_j})$ is the diagonal matrix formed by the components of $x(\beta_{k_j})$. Thus, for any i with $1 \leqslant i \leqslant n$,

$$\begin{aligned} 0 &\leqslant (u^i)^\top Qu^i + \beta_{k_j}(u^i)^\top((X(\beta_{k_j}) - L)^{-1} + (U - X(\beta_{k_j}))^{-1})u^i \\ &= q_{ii} + \beta_{k_j}\left(\tfrac{1}{x_i(\beta_{k_j})-l_i} + \tfrac{1}{u_i - x_i(\beta_{k_j})}\right), \end{aligned} \tag{2.5}$$

where u^i is the ith unit vector of R^n. From $q_{ii} < 0$ and (2.5), we derive that as $j \to \infty$, $x_i(\beta_{k_j})$ must approach either l_i or u_i because β_{k_j} goes to zero. Therefore, v^* is a vertex of B. The theorem follows. □

THEOREM 3. *For $k = 1, 2, \cdots$, let x^k be a local minimum point of*

$$\min_{l\leqslant x\leqslant u} e(x, \beta_k).$$

Assume that $Qv + c \neq 0$ at any limit point v of x^k, $k = 1, 2, \cdots$. If all the diagonal entries of Q are negative, every limit point of x^k, $k = 1, 2, \cdots$, is a local minimum point of (2.1).

Proof. Since x^k, $k = 1, 2, \cdots$, are contained in the bounded set B, we can extract a convergent subsequence. Let x^{k_q}, $q = 1, 2, \cdots$, be a convergent subsequence of x^k, $k = 1, 2, \cdots$. Assume that $\lim_{q\to\infty} x^{k_q} = v$. Let X_{k_q} be the diagonal matrix formed by the components of x^{k_q}. Since x^{k_q} is a local minimum point of $\min_{l\leqslant x\leqslant u} e(x, \beta_{k_q})$, hence, the Hessian matrix of $e(x, \beta_{k_q})$ at x^{k_q},

$$Q + \beta_{k_q}((X_{k_q} - L)^{-1} + (U - X_{k_q})^{-1}),$$

is positive semidefinite. Thus, for any i with $1 \leqslant i \leqslant n$,

$$\begin{aligned} 0 &\leqslant (u^i)^\top Qu^i + \beta_{k_q}(u^i)^\top((X_{k_q} - L)^{-1} + (U - X_{k_q})^{-1})u^i \\ &= q_{ii} + \beta_{k_q}\left(\tfrac{1}{x_i^{k_q}-l_i} + \tfrac{1}{u_i - x_i^{k_q}}\right), \end{aligned} \tag{2.6}$$

where u^i is the ith unit vector of R^n. From $q_{ii} < 0$ and (2.6), we obtain that as $q \to \infty$, $x_i^{k_q}$ must approach either l_i or u_i because β_{k_q} goes to zero. Therefore, v is an extreme point of B.

Since x^{k_q} is a local minimum point of

$$\min_{l \leqslant x \leqslant u} e(x, \beta_{k_q}),$$

from the first-order necessary optimality condition, we obtain

$$Qx^{k_q} + c = -\beta_{k_q} \left(\ln \frac{x_1^{k_q} - l_1}{u_1 - x_1^{k_q}}, \ln \frac{x_2^{k_q} - l_2}{u_2 - x_2^{k_q}}, \cdots, \ln \frac{x_n^{k_q} - l_n}{u_n - x_n^{k_q}} \right)^\top.$$

Hence,

$$\lim_{q \to \infty} -\beta_{k_q} (\ln \frac{x_1^{k_q} - l_1}{u_1 - x_1^{k_q}}, \ln \frac{x_2^{k_q} - l_2}{u_2 - x_2^{k_q}}, \cdots, \ln \frac{x_n^{k_q} - l_n}{u_n - x_n^{k_q}})^\top = Qv + c \neq 0. \tag{2.7}$$

Let x be an arbitrary interior point of B. Then,

$$(x - x^{k_q})^\top (Qx^{k_q} + c) = -\sum_{i=1}^{n} \beta_{k_q} (x_i - x_i^{k_q}) \ln \frac{x_i^{k_q} - l_i}{u_i - x_i^{k_q}}.$$

Note that v is an extreme point of B. Consider $v_i = l_i$. We have $x_i - v_i > 0$ and $\lim_{q \to \infty} x_i^{k_q} = l_i$. Thus, when q is sufficiently large,

$$\beta_{k_q} (x_i - v_i) \ln \frac{x_i^{k_q} - l_i}{u_i - x_i^{k_q}} < 0.$$

Consider $v_i = u_i$. We have $x_i - v_i < 0$ and $\lim_{q \to \infty} x_i^{k_q} = u_i$. Thus, when q is sufficiently large,

$$\beta_{k_q} (x_i - v_i) \ln \frac{x_i^{k_q} - l_i}{u_i - x_i^{k_q}} < 0.$$

From (2.7), we obtain that at least one of

$$\lim_{q \to \infty} \beta_{k_q} \ln \frac{x_i^{k_q} - l_i}{u_i - x_i^{k_q}},$$

$i = 1, 2, \cdots, n$, is not equal to zero. Therefore, at least one of

$$(x_i - v_i) \lim_{q \to \infty} \beta_{k_q} \ln \frac{x_i^{k_q} - l_i}{u_i - x_i^{k_q}},$$

$i = 1, 2, \cdots, n$, is negative, and all of them are not positive. Hence,

$$
\begin{aligned}
(x-v)^\top(Qv+c) &= \lim_{q\to\infty}(x-x^{k_q})^\top(Qx^{k_q}+c)\\
&= -\sum_{i=1}^n \lim_{q\to\infty}\beta_{k_q}(x_i - x_i^{k_q})\ln\frac{x_i^{k_q}-l_i}{u_i-x_i^{k_q}}\\
&= -\sum_{i=1}^n (x_i-v_i)\lim_{q\to\infty}\beta_{k_q}\ln\frac{x_i^{k_q}-l_i}{u_i-x_i^{k_q}}\\
&> 0.
\end{aligned}
$$

We have obtained that for any interior point x of B,

$$0 < (x-v)^\top(Qv+c). \tag{2.8}$$

Observe that

$$
\begin{aligned}
f(x) - f(v) &= \tfrac{1}{2}x^\top Qx + c^\top x - \tfrac{1}{2}v^\top Qv - c^\top v\\
&= (x-v)^\top(Qv+c) + \tfrac{1}{2}(x-v)^\top Q(x-v).
\end{aligned}
$$

Then, when x is sufficiently close to v, from (2.8), we obtain that

$$f(x) - f(v) > 0$$

since $\frac{1}{2}(x-v)^\top Q(x-v)$ goes to zero two times as fast as $(x-v)^\top(Qv+c)$ if x approaches v. This implies that v is a local minimum point of (2.1). The theorem follows. □

This theorem means that at least a local minimum point of (2.1) can be obtained if we are able to generate a local minimum point of the barrier function for a sequence of decreasing values of the barrier parameter with zero limit.

In the following we demonstrate through a two-dimensional example that the barrier term may help us obtain a global or near global optimal solution.

EXAMPLE 1. *Consider*

$$
\begin{aligned}
\min\ & f(x) = \tfrac{1}{2}(x_1, x_2)\begin{pmatrix} -83.75 & 28.34 \\ 28.34 & -48.28 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (17.72, 15.22)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\\
\textit{s/t}\ & 0 \le x_1 \le 1,\ 0 \le x_2 \le 1,
\end{aligned} \tag{2.9}
$$

which is generated randomly. From Figure 2, one can see that (2.9) has four local minimum points, which are $(1, 1)$, $(1, 0)$, $(0, 1)$, *and* $(0, 0)$. *The unique global minimum point is* $(1, 0)$. *Using (2.3), we obtain*

$$
\begin{aligned}
e(x,\beta) = &\ \tfrac{1}{2}(x_1, x_2)\begin{pmatrix} -83.75 & 28.34 \\ 28.34 & -48.28 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (17.72, 15.22)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\\
&+\beta(x_1\ln x_1 + (1-x_1)\ln(1-x_1) + x_2\ln x_2 + (1-x_2)\ln(1-x_2)).
\end{aligned}
$$

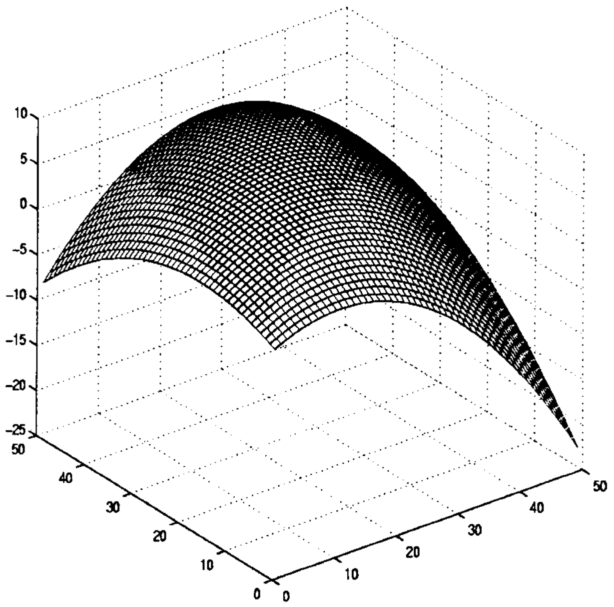

Figure 1. The surface of $f(x)$ over $[0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 1]$

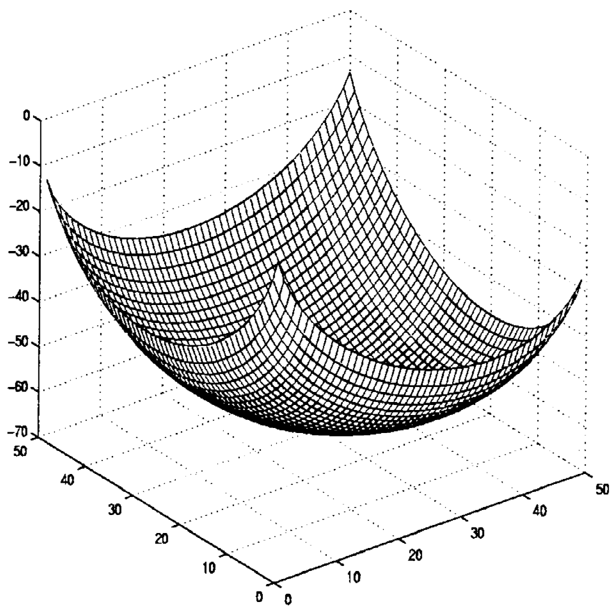

Figure 2. The surface of $e(x, 50)$ over $[0 < x_1 < 1,\ 0 < x_2 < 1]$

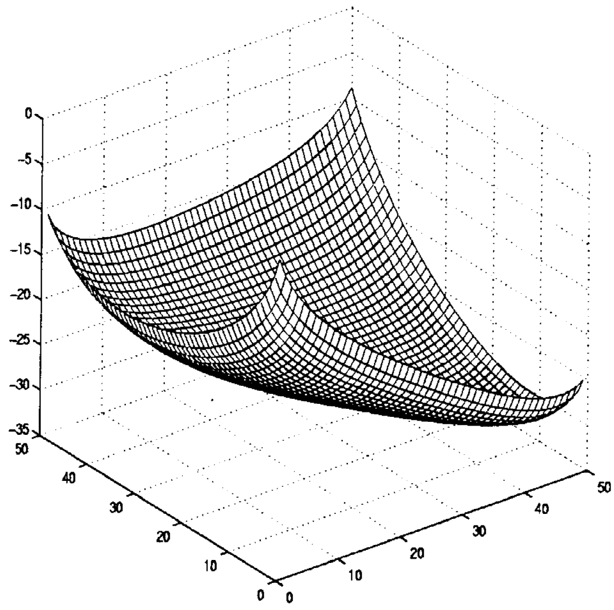

Figure 3. The surface of $e(x, 25)$ over $[0 < x_1 < 1,\ 0 < x_2 < 1]$

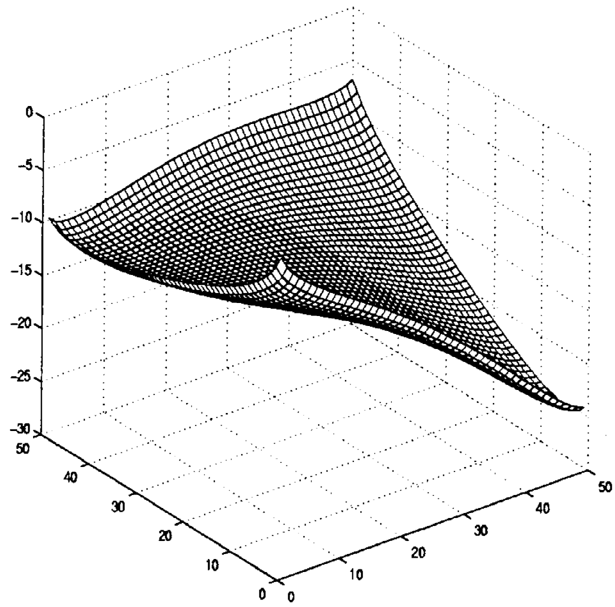

Figure 4. The surface of $e(x, 15)$ over $[0 < x_1 < 1,\ 0 < x_2 < 1]$

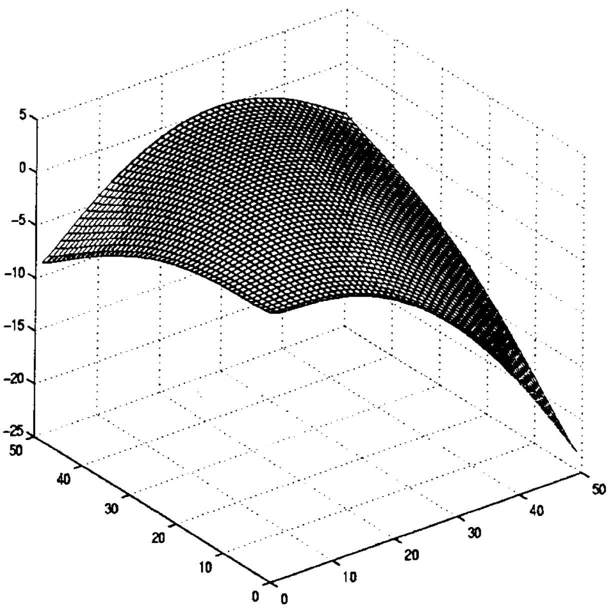

Figure 5. The surface of $e(x, 5)$ over $[0 < x_1 < 1,\ 0 < x_2 < 1]$

Figures 2–5 are the surfaces of $e(x, \beta)$ *with* $\beta = 50$, $\beta = 25$, $\beta = 15$, *and* $\beta = 5$, *respectively. When* β *equals* 25 *and* 15, *one can see from Figures 3 and 4 that* $e(x, \beta)$ *has only one local minimum point, which is near the global minimum point* $(1, 0)$.

This example show that $e(x, \beta)$ deforms from a strictly convex function to the objective function as β decreases from a sufficiently large positive number to zero and that there seemingly exists an interval of β such that each local minimum point of $e(x, \beta)$ is in a neighborhood of a global or near global minimum point when β is in the interval. Effectiveness of the barrier function method depends on existence of such an interval.

3. The Method

For any given $\beta > 0$, consider the first-order necessary optimality condition,

$$\frac{\partial e(x, \beta)}{\partial x_i} = 0, \tag{3.10}$$

$i = 1, 2, \cdots, n$. From (3.10), we obtain

$$x_i = \frac{u_i + l_i \exp(\frac{1}{\beta}\frac{\partial f(x)}{\partial x_i})}{1 + \exp(\frac{1}{\beta}\frac{\partial f(x)}{\partial x_i})},$$

$i = 1, \cdots, n$. For convenience of the following discussions, let

$$d_i(x) = \frac{u_i + l_i\gamma_i(x)}{1 + \gamma_i(x)},$$

$i = 1, \cdots, n$, and

$$d(x) = (d_1(x), \cdots, d_n(x))^\top,$$

where

$$\gamma_i(x) = \exp(\frac{1}{\beta}\frac{\partial f(x)}{\partial x_i}).$$

The following lemma shows that for any given $\beta > 0$, when $l < x < u$, $d(x) - x$ is a descent direction of $e(x, \beta)$.

LEMMA 2. *Assume* $l < x < u$. *For* $i = 1, \cdots, n$,
when $d_i(x) - x_i > 0$,

$$\frac{\partial e(x, \beta)}{\partial x_i} < 0,$$

when $d_i(x) - x_i < 0$,

$$\frac{\partial e(x, \beta)}{\partial x_i} > 0,$$

and when $d_i(x) - x_i = 0$,

$$\frac{\partial e(x, \beta)}{\partial x_i} = 0.$$

When $d(x) - x \neq 0$,

$$\nabla_x e(x, \beta)^\top (d(x) - x) < 0.$$

Proof.

1. Consider $d_i(x) - x_i < 0$. We have

$$\frac{u_i + l_i\gamma_i(x)}{1 + \gamma_i(x)} < x_i.$$

Thus we obtain

$$1 < \gamma_i(x)\frac{x_i - l_i}{u_i - x_i}. \tag{3.11}$$

Taking the natural logarithm to both sides of (3.11), we get

$$0 < \frac{1}{\beta}\frac{\partial f(x)}{\partial x_i} + \ln\frac{x_i - l_i}{u_i - x_i}. \tag{3.12}$$

Multiplying $\beta > 0$ to both sides of (3.12), we obtain

$$0 < \frac{\partial f(x)}{\partial x_i} + \beta \ln \frac{x_i - l_i}{u_i - x_i} = \frac{\partial e(x, \beta)}{\partial x_i}.$$

Therefore, when $d_i(x) - x_i < 0$,

$$\frac{\partial e(x, \beta)}{\partial x_i} > 0.$$

2. Consider $d_i(x) - x_i > 0$. We have

$$\frac{u_i + l_i \gamma_i(x)}{1 + \gamma_i(x)} > x_i.$$

Thus we obtain

$$1 > \gamma_i(x) \frac{x_i - l_i}{u_i - x_i}. \tag{3.13}$$

Taking the natural logarithm to both sides of (3.13), we get

$$0 > \frac{1}{\beta} \frac{\partial f(x)}{\partial x_i} + \ln \frac{x_i - l_i}{u_i - x_i}. \tag{3.14}$$

Multiplying $\beta > 0$ to both sides of (3.14), we obtain

$$0 > \frac{\partial f(x)}{\partial x_i} + \beta \ln \frac{x_i - l_i}{u_i - x_i} = \frac{\partial e(x, \beta)}{\partial x_i}.$$

Therefore, when $d_i(x) - x_i > 0$,

$$\frac{\partial e(x, \beta)}{\partial x_i} < 0.$$

3. Consider $d_i(x) - x_i = 0$. We have

$$\frac{u_i + l_i \gamma_i(x)}{1 + \gamma_i(x)} = x_i.$$

Thus we obtain

$$1 = \gamma_i(x) \frac{x_i - l_i}{u_i - x_i}. \tag{3.15}$$

Taking the natural logarithm to both sides of (3.15), we get

$$0 = \frac{1}{\beta} \frac{\partial f(x)}{\partial x_i} + \ln \frac{x_i - l_i}{u_i - x_i}. \tag{3.16}$$

Multiplying $\beta > 0$ to both sides of (3.16), we obtain

$$0 = \frac{\partial f(x)}{\partial x_i} + \beta \ln \frac{x_i - l_i}{u_i - x_i} = \frac{\partial e(x, \beta)}{\partial x_i}.$$

Therefore, when $d_i(x) - x_i = 0$,

$$\frac{\partial e(x, \beta)}{\partial x_i} = 0.$$

Observe that

$$\nabla_x e(x,\beta)^\top (d(x)-x) = \sum_{i=1}^{n} \frac{\partial e(x,\beta)}{\partial x_i}(d_i(x)-x_i).$$

The lemma follows. □

Note that for any x with $l < x < u$, $d(x) - x = 0$ if and only if $\nabla_x e(x,\beta) = 0$. We remark that $d(x) - x$ has a desired property that when searching for a point in $d(x) - x$, the box constraints are always satisfied automatically if the step length is a number between zero and one.

Based on the descent direction, $d(x) - x$, we have developed a method for approximating a solution of the problem (2.1). The idea of the method is as follows:

Let β_q, $q = 1, 2, \cdots$, be any given sequence of positive numbers such that $\beta_1 > \beta_2 > \cdots$ and $\lim_{q\to\infty} \beta_q = 0$. The value of β_1 should be sufficiently large so that $e(x,\beta_1)$ is strictly convex over $l \leqslant x \leqslant u$. Let x^0 be an arbitrary interior point of B. For $q = 1, 2, \cdots$, starting at x^{q-1}, we employ $d(x) - x$ as the descent direction to search for an interior point $x^q \in B$ satisfying $d(x^q) - x^q = 0$.

The method can be stated as follows.

Step 0: Let θ be a number in $(0, 1)$, which should be close to one. Choose x^0 to be an arbitrary point satisfying $l < x^0 < u$, and β to be an arbitrary positive number satisfying that $e(x,\beta)$ is strictly convex over $l \leqslant x \leqslant u$. Let $k = 0$ and go to Step 1.

Step 1: Compute

$$d_i(x^k) = \frac{u_i + l_i\gamma_i(x^k)}{1+\gamma_i(x^k)},$$

$i = 1, \cdots, n$. *Go to Step 2.*

Step 2: If $\|d(x^k)-x^k\|$ is less than some given tolerance, either the method terminates when β is small enough (e.g., a vertex minimum point can be recovered from x^k if (2.1) has a vertex solution), or let $\beta = \theta\beta$ and go to Step 1. Otherwise, do as follows: Compute

$$x^{k+1} = x^k + \mu_k(d(x^k) - x^k), \tag{3.17}$$

where μ_k is a number in $[0, 1]$ satisfying

$$e(x^{k+1},\beta) = \min_{\mu\in[0,1]} e(x^k + \mu(d(x^k) - x^k), \beta).$$

Let $k = k + 1$ and go to Step 1.

Note that an exact solution of

$$\min_{\mu\in[0,1]} e(x^k + \mu(d(x^k) - x^k), \beta)$$

is not required in the implementation of the method, and an approximate solution will do. One can find several ways to determine μ_k in Minoux (1986). We remark that the method is insensitive to the starting point x^0 since the barrier function is strictly convex at the beginning of the method. The following theorem shows that when β is a given positive value, the method converges to a stationary point of $e(x, \beta)$.

THEOREM 4. *For a given $\beta > 0$, every limit point of x^k, $k = 1, 2, \cdots$, generated by the iterative procedure (3.17) is a stationary point of $e(x, \beta)$.*

Proof. Recall that $\gamma_i(x) = \exp(\frac{1}{\beta}\frac{\partial f(x)}{\partial x_i})$. Let

$$\gamma_i^{\min} = \min_{x \in B} \gamma_i(x) \text{ and } \gamma_i^{\max} = \max_{x \in B} \gamma_i(x).$$

Since $\frac{\partial f(x)}{\partial x_i}$ is continuous on B, we obtain that $0 < \gamma_i^{\min} < \infty$ and $0 < \gamma_i^{\max} < \infty$. Consider

$$h(w) = \frac{s + tw}{1 + w}$$

with $s > t$. We have

$$h'(w) = \frac{t - s}{(1 + w)^2} < 0.$$

Thus, h is a decreasing function of w. Letting $w = \gamma_i(x)$, we get

$$d_i(x) = \frac{u_i + l_i\gamma_i(x)}{1 + \gamma_i(x)} = \frac{u_i + l_i w}{1 + w},$$

which has the same form as $h(w)$. Therefore, for any $x \in B$,

$$l_i < \frac{u_i + l_i\gamma_i^{\max}}{1 + \gamma_i^{\max}} \leqslant d_i(x) \leqslant \frac{u_i + l_i\gamma_i^{\min}}{1 + \gamma_i^{\min}} < u_i.$$

Let

$$x_i^{\min} = \min\{x_i^0, \frac{u_i + l_i\gamma_i^{\max}}{1 + \gamma_i^{\max}}\} \text{ and } x_i^{\max} = \max\{x_i^0, \frac{u_i + l_i\gamma_i^{\min}}{1 + \gamma_i^{\min}}\}.$$

Because $l_i < x_i^0 < u_i$, we have $l_i < x_i^{\min}$ and $x_i^{\max} < u_i$. Let

$$x^{\min} = (x_1^{\min}, \cdots, x_n^{\min})^\top \text{ and } x^{\max} = (x_1^{\max}, \cdots, x_n^{\max})^\top.$$

Using $x^{k+1} = x^k + \mu_k(d(x^k) - x^k)$ and $0 \leqslant \mu_k \leqslant 1$, one can easily obtain that x^k, $k = 1, 2, \cdots$, satisfy

$$l < x^{\min} \leqslant x^k \leqslant x^{\max} < u.$$

Thus, according to Lemma 2, $d(x^k) - x^k$ is a descent direction of $e(x, \beta)$ when $d(x^k) - x^k \neq 0$.

Let $X = \{x \mid x^{\min} \leqslant x \leqslant x^{\max}\}$ and $\Omega = \{x \in X \mid \nabla_x e(x, \beta) = 0\}$. For any $x \in X$, let

$$A(x) = \left\{ x + \mu^*(d(x) - x) \;\middle|\; \begin{array}{l} \mu^* \in [0, 1],\ e(x + \mu^*(d(x) - x), \beta) \\ = \min_{\mu \in [0,1]} e(x + \mu(d(x) - x), \beta) \end{array} \right\}.$$

In the following we prove that $A(x)$ is closed at every point $x \in X \backslash \Omega$.

Let $\bar{x}$ be an arbitrary point of $X \backslash \Omega$. Let $x^q \in X \backslash \Omega$, $q = 1, 2, \cdots$, be a sequence convergent to $\bar{x}$, and $y^q \in A(x^q)$, $q = 1, 2, \cdots$, a sequence convergent to $\bar{y}$. To prove that $A(\bar{x})$ is closed, we only need to show $\bar{y} \in A(\bar{x})$. Since $\nabla_x e(x^q, \beta) \neq 0$ and $\nabla_x e(\bar{x}, \beta) \neq 0$, we obtain from Lemma 2 that $d(x^q) - x^q \neq 0$ and $d(\bar{x}) - \bar{x} \neq 0$. Observe that $d(x)$ is continuous. Thus, $d(x^q)$ converges to $d(\bar{x})$ as $q \to \infty$. Since $y^q \in A(x^q)$, hence, there is some number $\mu_q^* \in [0, 1]$ satisfying

$$y^q = x^q + \mu_q^*(d(x^q) - x^q).$$

From $d(x^q) - x^q \neq 0$ we obtain that

$$\mu_q^* = \frac{\|y^q - x^q\|}{\|d(x^q) - x^q\|},$$

and as $q \to \infty$,

$$\mu_q^* \to \bar{\mu}^* = \frac{\|\bar{y} - \bar{x}\|}{\|d(\bar{x}) - \bar{x}\|}$$

with $\bar{\mu}^* \in [0, 1]$. Therefore,

$$\bar{y} = \bar{x} + \bar{\mu}^*(d(\bar{x}) - \bar{x}).$$

Furthermore, since $y^q \in A(x^q)$, we have

$$e(y^q, \beta) \leqslant e(x^q + \mu(d(x^q) - x^q), \beta)$$

for any $\mu \in [0, 1]$. It implies that

$$e(\bar{y}, \beta) \leqslant e(\bar{x} + \mu(d(\bar{x}) - \bar{x}), \beta)$$

for any $\mu \in [0, 1]$, which proves that

$$e(\bar{y}, \beta) = \min_{\mu \in [0,1]} e(\bar{x} + \mu(d(\bar{x}) - \bar{x}), \beta).$$

According to the definition of $A(x)$, it follows that $\bar{y} \in A(\bar{x})$.

Since X is bounded and $x^k \in X$, $k = 1, 2, \cdots$, we can extract a convergent subsequence from the sequence, x^k, $k = 1, 2, \cdots$. Let x^{k_j}, $j = 1, 2, \cdots$, be a convergent subsequence of the sequence, x^k, $k = 1, 2, \cdots$. Let x^* be the limit point of the subsequence. We show $x^* \in \Omega$ in the following. Clearly, as $k \to \infty$, $e(x^k, \beta)$ converges to $e(x^*, \beta)$ since $e(x, \beta)$ is continuous on B and $e(x^{k+1}, \beta) < e(x^k, \beta)$, $k = 0, 1, \cdots$. Consider the sequence, x^{k_j+1}, $j = 1, 2, \cdots$. Note that $x^{k_j+1} = x^{k_j} + \mu_{k_j}(d(x^{k_j}) - x^{k_j})$ and

$$e(x^{k_j+1}, \beta) = \min_{\mu \in [0,1]} e(x^{k_j} + \mu(d(x^{k_j}) - x^{k_j}), \beta).$$

According to the definition of $A(x)$, we have $x^{k_j+1} \in A(x^{k_j})$. Since x^{k_j+1}, $j = 1, 2, \cdots$, are bounded, we can extract a convergent subsequence from the sequence, x^{k_j+1}, $j = 1, 2, \cdots$. Let x^{k_j+1}, $j \in K$, be a convergent subsequence extracted from the sequence, x^{k_j+1}, $j = 1, 2, \cdots$. Let $x^{\#}$ be the limit point of the subsequence, x^{k_j+1}, $j \in K$. Suppose that $x^* \notin \Omega$. Since $A(x^*)$ is closed, we have $x^{\#} \in A(x^*)$. Thus,

$$e(x^{\#}, \beta) < e(x^*, \beta),$$

which contradicts that $e(x^k, \beta)$ converges as $k \to \infty$. Therefore, $x^* \in \Omega$. The theorem follows. □

4. Numerical Results

In this section we use the method to approximate solutions of a number of nonconvex quadratic programming problems with box constraints. The method is programmed in MATLAB. To determine μ_k in the method, we employ the Armijo-type rule, which can be stated as follows:
Let δ and ν be any two given numbers in $(0, 1)$. Choose m_k to be the smallest nonnegative integer satisfying

$$e(x^k + \nu^{m_k}(d(x^k) - x^k), \beta) \leqslant e(x^k, \beta) + \nu^{m_k}\delta(d(x^k) - x^k)^\top \nabla_x e(x^k, \beta).$$

Let $\mu_k = \nu^{m_k}$.

In our implementation of the method, $\delta = 0.6$ and $\nu = 0.4$. Numerical results are as follows.

EXAMPLE 2. *Find a global minimum point of*

$$\begin{aligned} &\min \;\; f(x) = c^\top x - \tfrac{1}{2} x^\top A A^\top x \\ &\textit{subject to} \;\; 0 \leqslant x_i \leqslant 1, \;\; i = 1, \cdots, n, \end{aligned} \tag{4.18}$$

where A *is an* $n \times m$ *matrix with entries being random numbers in* $[-1, 1]$ *and* c *is a vector with components being random numbers in* $[-1, 1]$. *Initially,* $\beta = 200$, *and is decreased by a factor* $\theta = 0.95$ *when* $\|d(x^k) - x^k\|_2 < 0.01$. *The method terminates when* $\beta < 0.1$. *A vertex solution is given by* $z^* = round(x^k)$. *Starting at an arbitrary point* x^0 *satisfying* $0 < x_i^0 < 1$, $i = 1, \cdots, n$, *the method always generates a global minimum point for our tests generated randomly.*

$\|x^k - z^*\|_2$	No. of Iterations	$\|x^k - z^*\|_2$	No. of Iterations
6.0340E-3	181	7.4854E-3	166
9.1815E-3	191	9.4808E-3	178
6.8765E-3	197	6.5753E-3	185
9.2087E-3	193	5.8108E-3	174
7.3018E-3	186	7.1863E-3	177

Figure 6. Numerical results

$\|x^k - z^*\|_2$	No. of Iterations	$\|x^k - z^*\|_2$	No. of Iterations
6.2459E-3	171	5.8474E-3	187
6.1861E-3	199	6.9777E-3	185
9.1458E-3	208	8.5286E-3	167
6.2287E-3	176	6.4298E-3	178
7.3265E-3	166	5.5457E-3	162

Figure 7. Numerical results

For $n = 20$ *and* $m = 25$*, ten randomly generated problems have been solved. Numerical results are given in Figure 6.*

For $n = 20$ *and* $m = 30$*, ten randomly generated problems have been solved. Numerical results are given in Figure 7.*

EXAMPLE 3. *Find a global minimum point of*

$$\begin{aligned} &\min \ f(x) = c^\top x + \tfrac{1}{2} x^\top Q x \\ &\text{subject to } -1 \leqslant x_i \leqslant 1, \ i = 1, \cdots, n, \end{aligned} \tag{4.19}$$

where Q *is a symmetric matrix and* $c = -(Q - \mu I)z^*$ *with* μ *being the smallest eigenvalue of* Q *and* $z^* = (1, -1, \cdots, 1, -1)^\top$*. The way of generating this problem is given in Pardalos (1991). It is easy to see that* z^* *is a global minimum point of (4.19). Initially,* $\beta = 100$*, and is decreased by a factor* $\theta = 0.95$ *when* $\|d(x^k) - x^k\|_2 < 0.01$*. The method terminates when* $\beta < 5$*. Starting at an arbitrary point* x^0 *satisfying* $0 < x_i^0 < 1$*,* $i = 1, \cdots, n$*, the method always generates the global minimum point for our tests generated randomly.*

When the entries of the upper triangular part of Q *are numbers taken randomly from* $[-5, 5]$*, numerical results are given in Figure 8.*

n	$\|x^k - z^*\|_2$	No. of Iterations	n	$\|x^k - z^*\|_2$	No. of Iterations
100	3.2193E-3	285	200	2.9778E-3	249
300	6.3524E-4	249	400	1.3012E-3	225
500	2.2677E-3	218	600	2.7916E-4	206
700	2.3449E-3	204	800	1.1183E-3	197
900	1.0648E-3	193	1000	1.0889E-3	190

Figure 8. Numerical results

n	$\|x^k - z^*\|_2$	No. of Iterations	n	$\|x^k - z^*\|_2$	No. of Iterations
100	1.5037E-3	142	200	2.2577E-3	117
300	2.1031E-3	93	400	2.6111E-3	79
500	1.2193E-3	64	600	1.1041E-3	61
700	1.0389E-3	54	800	5.1412E-4	44
900	4.3362E-3	38	1000	2.7580E-3	33

Figure 9. Numerical results

n	$\|x^k - z^*\|_2$	No. of Iterations	n	$\|x^k - z^*\|_2$	No. of Iterations
50	3.8401E-3	868	100	3.7658E-3	1536
150	3.0655E-3	6099	200	1.4486E-3	8016
250	2.5742E-3	3879	300	2.8251E-3	7850

Figure 10. Numerical results

When the entries of the upper triangular part of Q are numbers taken randomly from $[-15, 15]$, *numerical results are given in Figure 9.*

EXAMPLE 4. *Find a global minimum point of*

$$\begin{aligned} &\min\ f(x) = -(n-1)\sum_{i=1}^{n} x_i - \frac{1}{n}\sum_{i=1}^{\frac{n}{2}} x_i + 2\sum_{i<j} x_i x_j \\ &\textit{subject to}\ \ 0 \leqslant x_i \leqslant 1,\ \ i = 1, \cdots, n, \end{aligned} \tag{4.20}$$

where n is an even positive integer. This problem is given in Pardalos (1991) and has an exponential number of local minimum points. The unique global minimum point is $z^* = (1, \cdots, 1, 0, \cdots, 0)^\top$, which has $n/2$ components of one. This problem has been considered as a benchmark problem for testing effectiveness and efficiency of algorithms for quadratic zero-one programming problems. We have used the method to solve (4.20) up to $n = 300$. *Initially,* $\beta = 100$, *and is decreased by a factor* $\theta = 0.95$ *when* $\|d(x^k) - x^k\|_2 < 0.01$. *The method terminates when* $\beta < 0.1$. *Starting at an arbitrary point x^0 satisfying* $0 < x_i^0 < 1$, $i = 1, \cdots, n$, *the method always generates the global minimum point. Numerical results are given in Figure 10.*

EXAMPLE 5. *Let $G = (V, E)$ be an undirected graph, where* $V = \{1, 2, \cdots, n\}$ *is the node set of G and E is the edge set of G. Let (i, j) denote an edge between node i and node j. Let*

$$A_G = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{n1} \\ a_{21} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

represent the adjacency matrix of G, where $a_{ii} = 0$ for $i = 1, \cdots, n$, and

$$a_{ij} = \begin{cases} 1 & \textit{if } (i, j) \in E, \\ 0 & \textit{otherwise,} \end{cases}$$

for any $i \neq j$. Observe that a graph is completely determined by its adjacency matrix A_G.

A complement graph of G, denoted by $\bar{G}$, is the graph $\bar{G} = (V, \bar{E})$ with

$$\bar{E} = \{(i, j) \mid (i, j) \notin E \text{ and } i \neq j\}.$$

Let

$$A_{\bar{G}} = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} & \cdots & \bar{a}_{n1} \\ \bar{a}_{21} & \bar{a}_{22} & \cdots & \bar{a}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{n1} & \bar{a}_{n2} & \cdots & \bar{a}_{nn} \end{pmatrix}$$

represent the adjacency matrix of $\bar{G}$, where $\bar{a}_{ii} = 0$ for $i = 1, \cdots, n$, and $\bar{a}_{ij} = 1 - a_{ij}$ for any $i \neq j$.

Let S be a subset of V. A subgraph of G with the node set S, denoted by $G(S)$, is the graph $G(S) = (S, E(S))$ with

$$E(S) = \{(i, j) \mid (i, j) \in E \text{ and } i, j \in S\}.$$

A graph G is complete if $(i, j) \in E$ for any $i \neq j$. A clique of G is a subset C of V such that $G(C)$ is complete. A maximum clique of G is a clique that has the maximum cardinality. The maximum clique problem seeks for a clique of the maximum cardinality.

It can be found in Pardalos and Rodgers (1992) that finding a maximum clique of a graph $G = (V, E)$ is equivalent to solving

$$\begin{aligned} &\min \quad f(x) = x^\top Q x \\ &\text{subject to} \quad 0 \leqslant x \leqslant 1, \end{aligned} \tag{4.21}$$

where $Q = A_{\bar{G}} - I$ with I being the identity matrix. Observe that the problem has a vertex global minimum point since all the diagonal entries of Q are negative. The method has been used to find a maximum clique of a graph $G = (V, E)$, whose adjacency matrix A_G is randomly generated with the following procedure:

Let p be a number in $(0, 1)$. For $i = 1, \cdots, n$, and $j = i + 1, \cdots, n$, choose a number $\alpha \in (0, 1)$ according to the uniform probability distribution, and let $a_{ij} = 1$ and $a_{ji} = 1$ if $\alpha \leqslant p$, and $a_{ij} = 0$ and $a_{ji} = 0$ if $\alpha > p$.

Initially, $\beta = 100$, and is decreased by a factor $\theta = 0.95$ when $\|d(x^k) - x^k\|_2 < 0.005$. The method terminates when $\beta < 0.1$. Let $z^* = round(x^k)$. A clique is given by $\{i \mid z_i^* = 1\}$. Starting at an arbitrary point x^0 satisfying $0 < x_i^0 < 1$, The Barrier Function Method

n	$\|x^k - z^*\|_2$	No. of Iterations	n	$\|x^k - z^*\|_2$	No. of Iterations
20	1.7055E-3	356	40	1.6980E-3	528
60	1.9405E-3	577	80	1.8260E-3	730
100	1.9966E-3	721	120	2.0028E-3	828

The Branch and Bound Algorithm

n	No. of Iterations	n	No. of Iterations
20	131	40	2439
60	18252	80	123335
100	420805	120	2196651

Figure 11. Numerical results

The Barrier Function Method

n	$\|x^k - z^*\|_2$	No. of iterations	n	$\|x^k - z^*\|_2$	No. of Iterations
20	2.2014E-3	367	40	2.0519E-3	458
60	1.9995E-3	542	80	9.4121E-4	620
100	1.0053E-3	693	120	1.5981E-3	791

The Branch and Bound Algorithm

n	No. of Iterations	n	No. of Iterations
20	199	40	4677
60	75446	80	910336
100	8846769	120	38171597

Figure 12. Numerical results

$i = 1, \cdots, n$, the method always generates a maximum or near maximum clique for our tests generated randomly.

When $p = 0.7$, the method has successfully found a maximum clique for $n = 20, 40, 60, 80, 100, 120$. To verify the cliques generated by the method are the maximum cliques, we have used the branch and bound algorithm proposed in Carraghan and Pardalos (1990) to solve the same problems. Numerical results are given in Figure 11.

When $p = 0.8$, the method has successfully found a maximum clique for $n = 20, 40, 60, 80, 100, 120$. To verify the cliques generated by the method are the maximum cliques, we have used the branch and bound algorithm proposed in Carraghan and Pardalos (1990) to solve the same problems. Numerical results are given in Figure 12.

From these numerical results, one can see that the method seems effective and efficient. Although these numerical results show that the method always generates a global or near global minimum point, it is difficult to theoretically prove that the method converges to a global or near global minimum point even when the barrier parameter decreases at a sufficiently slow pace.

5. Conclusions

In this paper we have developed a barrier function method for approximating a solution of the nonconvex quadratic programming problem with box constraints. The preliminary numerical results show that the method seems effective and efficient. We have also presented a two-dimensional example to show that the barrier term may help us obtain a global or near global optimal solution. Although all the preliminary numerical results show that the method always finds a global or near global minimum point when the barrier parameter decreases at a sufficiently slow pace, it is difficult to theoretically prove that the method always generates a global or near global optimal solution. We remark that combining the barrier function method and a branch-and-bound algorithm may provide an efficient approach to solving the maximum clique problem.

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References

1. Barahona, F., Junger M., and Reinelt G., (1989). Experiments in quadratic 0-1 programming, *Math. Program.* 44: 127–137.
2. Carraghan R. and Pardalos, P.M. (1990). An exact algorithm for the maximum clique problem, *Oper. Res. Lett.* 9: 375–382.
3. Carter, M.W., (1984). The indefinite zero-one quadratic problem, *Discrete Appl. Math.* 7, 23–44.
4. De Angelis, P.L., Pardalos, P.M. and Toraldo G., (1997). Quadratic programming with box constraints, in I.M. Bomze, T. Csendes, R. Horst, P.M. Pardalos (eds.), Developments in Global Optimization, Kluwer Academic Publishers, 73–93.
5. Durbin, R. and Willshaw, D. (1987). An analog approach to the traveling salesman problem using an elastic net method, *Nature* 326: 689–691.
6. Fang, S.C., Rajasekera, J.R. and Tsao, H.S.J. (1997). *Entropy Optimization and Mathematical Programming*, Kluwer Academic Publishers.
7. Floudas, C.A. and Visweswaran, V. (1995). Quadratic optimization, in R. Horst and P.M. Pardalos (eds.), *Handbook of Global Optimization*, Kluwer Academic Publishers, 217–269.
8. Grippo, L. and Lucidi, S. (1991). A differentiable exact penalty function for bound constrained quadratic programming problems, *Optimization* 22: 557–578.
9. Gulati, V.P., Gupta, S.K. and Mital, A.K., (1984). Unconstrained quadratic bivalent programming problems, *European J. of Op. Res.* 15: 121–125.
10. Hammer, P.L. and Simeone, B. (1987). Order relations of variables in 0-1 programming, *Ann. Discrete Maths* 31: 83–112.
11. Hansen, P. (1979). Methods of nonlinear zero-one programming, *Annals of Discrete Maths* 5: 53–70.

12. Hopfield, J.J. (1984). Neurons with graded response have collective computational properties like those of two-state neurons, *Proc. Nat. Acad. Sci U.S.A.* 81: 3088–3092.
13. Minoux, M. (1986). Mathematical Programming: *Theory and Algorithms*, John Wiley & Sons.
14. Murty, K.G. and Kabadi, S.N., (1987). Some NP-complete problems in quadratic and nonlinear programming, *Math. Prog.* 39: 117–129.
15. Pardalos, P.M. (1991). Construction of test problems in quadratic bivalent programming, *ACM Transact Math Software* 17: 74–87.
16. Pardalos, P.M. and Jha, S., (1992). Complexity of uniqueness and local search in quadratic 0-1 programming, *Oper. Res. Lett.* 11: 119–123.
17. Pardalos, P.M. and Rodgers, G.P., (1990). Computational aspects of a branch and bound algorithm for quadratic zero-one programming, *Computing* 45: 131–144.
18. Pardalos, P.M. and Rodgers, G.P., (1992). A branch and bound algorithm for the maximum clique problem, *Comput. Oper. Res.* 19: 363–375.
19. Pardalos, P.M. and Rosen, J.B., (1987). Constrained Global Optimization: Algorithms and Applications, *Lecture Notes in Computer Sciences* 268, Springer.
20. Peterson, C. (1990). Parallel distributed approaches to combinatorial optimization problems – benchmark studies on T.S.P., *Neural Computation* 2: 261–270.
21. Poljak, S. and Wolkowicz, H., (1995). Convex relaxations of (0,1)-quadratic programming, *Maths of Oper. Res.* 20: 550–561.
22. Vavasis, S.A., (1991). On approximation algorithms for concave quadratic programming, in C.A. Floudas and P.M. Pardalos (eds.), *Recent Advances in Global Optimization*, Princeton University Press, 1992, 3–18.
23. Ye, Y., (1992). On affine scaling algorithms for nonconvex quadratic programming, *Math Program.* 56: 285–300.
24. Yuille, A.L. and Kosowsky, J.J. (1994). Statistical physics algorithms that converges, *Neural Computation* 6: 341–356.