Measuring the Efficiency of an Algorithm by the Worst Input

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A significant part of computer science is devoted to understanding the power of the RAM model in solving specific problems, that is, what would be a "fastest" algorithm that can produce a correct output on every input to the underlying problem?

But how do we measure "fast"? One approach—the one we follow in this course—is to look at the algorithm's cost on the worst input, as we will formalize in this lecture.

Cost on the Worst Input

Define $\mathcal{I}_n$, where n is an integer, to be the set of all inputs to a problem that have the same **problem size** n.

Given an input $I \in \mathcal{I}_n$, the cost $X(I)$ of an algorithm $\mathcal{A}$ is the length of its execution on I.

- The **worst-case cost** of $\mathcal{A}$ under the problem size n is the maximum $X(I)$ of all $I \in \mathcal{I}_n$.
- The **worst expected cost** of $\mathcal{A}$ under the problem size n is the maximum $\mathbf{E}[X(I)]$ of all $I \in \mathcal{I}_n$.

Example: Dictionary Search

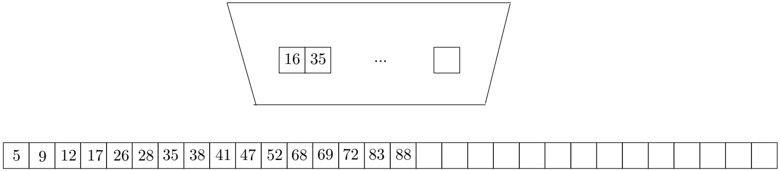
Problem Input: In the memory, a set S of n integers have been arranged in ascending order at the memory cells from address 1 to n. The value of n has been placed in Register 1 of the CPU. Another integer v has been placed in Register 2 of the CPU.

- n is the problem size.
- $\mathcal{I}_n$ is the set of all possible (S, v).

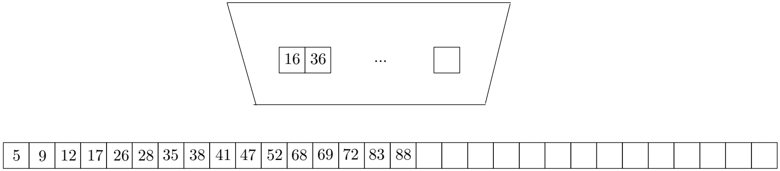
Goal: Determine whether v exists in S.

Example: Dictionary Search

A "yes"-input with $n = 16$



A "no"-input with $n = 16$



Example 1: Dictionary Search

The worst-case cost of the binary search algorithm is $O(\log n)$.

In other words, on any input in $\mathcal{I}_n$, the maximum number $f(n)$ of atomic operations performed by the algorithm must grow no faster than $\log_2 n$.

- Note: This does not mean $f(n) = \log_2 n$.
  "$f(n) = O(\log n)$" only says that $f(n)$ could be functions like $10(1 + \log_2 n)$, $352 \log_3 n$, $\sqrt{\log n} + 78 \log_2(n^{83})$, etc.

Example 2

Consider the following randomized algorithm:

```
/* $A$ is an array of size $n$ that contains at least one 0 */
1. do
2.     $r$ = RANDOM(1, $n$)
3. until $A[r] = 0$
4. return $r$
```

What is the expected cost of the algorithm? The answer is “it depends”:

- If all numbers in A are 0, the algorithm finishes in $O(1)$ time.
- If A has only one 0, the algorithm finishes in $O(n)$ expected time because
  - $A[r]$ has $1/n$ probability of being 0.
  - In expectation, we need to repeat n times to find the 0.

Example 2 (cont.)

```
/* $A$ is an array of size $n$ that contains at least one 0 */
1. do
2.     $r$ = RANDOM(1, $n$)
3. until $A[r] = 0$
4. return $r$
```

Worst-case cost of the algorithm $= \infty$
Worst expected cost of the algorithm $= O(n)$

Before finishing the lecture, we will tap into the power of randomization by witnessing a problem where randomized algorithms are provably faster than deterministic ones in expected cost.

Power of Randomization

Problem "Find-a-Zero": Let A be an array of n integers, among which half of them are 0. Design an algorithm to report an arbitrary position of A that contains a 0.

For example, suppose $A = (9, 18, 0, 0, 15, 0, 33, 0)$. An algorithm can report 3, 4, 6, or 8.

Power of Randomization

```
1. do
2.     $r$ = RANDOM(1, $n$)
3. until $A[r] = 0$
4. return $r$
```

The algorithm finishes in $O(1)$ expected time on **every input** A!

In contrast, any deterministic algorithm must probe at least $n/2$ integers of A in the worst case! In other words, any deterministic algorithm must have a worst case time of $\Theta(n)$—provably slower than the above randomized algorithm (in expectation).