Recursion (the Beginning)

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This lecture will introduce a technique called recursion for designing algorithms. Its principle is:

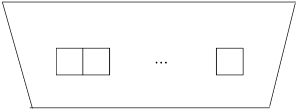
> When dealing with a subproblem (same problem but with a smaller input), consider it solved.

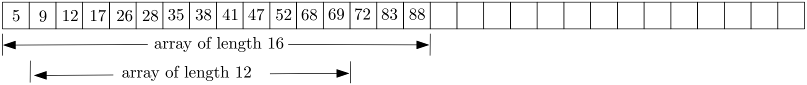
We will apply the technique to settle several problems in this course. Today, we will see two examples. In the first, we will re-discover binary search; in the second, we will design our first sorting algorithm.

Array

An **array** of **length** n is a sequence of n elements such that

- they are stored consecutively in memory (i.e., the first element is immediately followed by the second, and then by the third, and so on);
- every element occupies the same number of memory cells.





With the concept of array, we now redefine the dictionary search problem:

The Dictionary Search Problem (Redefined)

Problem Input:

A set S of n integers has been arranged in ascending order in an array of length n. You are given the value of n and another integer v inside the CPU.

Goal:

Design an algorithm to determine whether v exists in S.

Binary Search (Re-discovered)

1. Compare v to the middle element e of the array. If $v = e$, return "yes" and done.
2. Otherwise:
   2.1 If $v < e$, we have a **subproblem**: check if v is in the portion of the array before e;
   2.2 If $v > e$, we have a **subproblem**: check if v is in the portion of the array after e.

Considering the subproblem solved, we finish the algorithm.

Think: why does it work?

Analysis of Binary Search

Recursion allows us to analyze the running time in an elegant manner.

Define $f(n)$ to be the maximum running time of binary search on n elements. For $n = 1$, clearly:

$$f(1) = O(1)$$

For $n > 1$:

$$f(n) \leq O(1) + f(\lfloor n/2 \rfloor).$$

Analysis of Binary Search

So it remains to solve the recurrence (c_1, c_2 are constants whose values we do not care):

$$\begin{aligned} f(1) &= c_1 \\ f(n) &\leq c_2 + f(\lfloor n/2 \rfloor) \end{aligned}$$

Suppose, for now, that n is a power of 2. An easy way of doing so is the expansion method, which simply expands $f(n)$ all the way down:

$$\begin{aligned} f(n) &\leq c_2 + f(n/2) \\ &\leq c_2 + c_2 + f(n/2^2) \\ &\leq c_2 + c_2 + c_2 + f(n/2^3) \\ &\leq \underbrace{c_2 + ... + c_2}_{\log_2 n \text{ of them}} + f(1) \\ &= c_2 \cdot \log_2 n + c_1 = O(\log n). \end{aligned}$$

Analysis of Binary Search

We can deal with general n (not necessarily a power of 2) using a rounding approach. Let n' be the least power of 2 that is larger than n. It thus holds that $n' < 2n$ (otherwise, n' is not the least).

We then have:

$$
\begin{aligned}
f(n) &\le f(n') \\
&\le c_2 \cdot \log_2 n' + c_1 \text{ (proved earlier)} \\
&< c_2 \cdot \log_2(2n) + c_1 \\
&= c_2(1 + \log_2 n) + c_1 \\
&= c_2 \log_2 n + c_1 + c_2 = O(\log n).
\end{aligned}
$$

Next, we switch our attention to the sorting problem, which is a classical problem in computer science, and is worth several lectures' discussion.

The Sorting Problem

Problem Input:

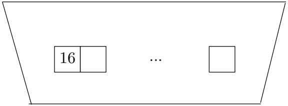
A set S of n integers is given in an array of length n. The value of n is inside the CPU (i.e., in a register).

Goal:

Produce an array that stores the elements of S in **ascending order**.

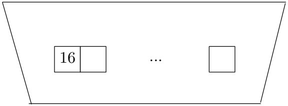
Example

Input:



38	28	88	17	26	41	72	83	69	47	12	68	5	52	35	9														

Output:



5	9	12	17	26	28	35	38	41	47	52	68	69	72	83	88														

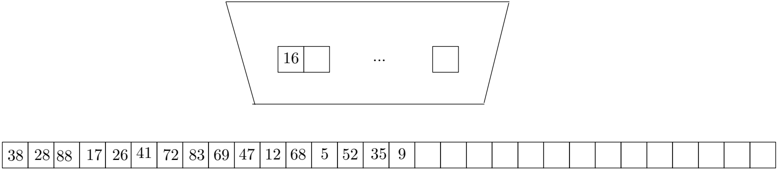
Selection Sort

1. Find the largest integer e_{max} in S.
2. Swap e_{max} with the last (i.e., n-th) element of the array (after which e_{max} is at the end of the array).
3. We now have a **subproblem**: sort the first $n - 1$ elements.

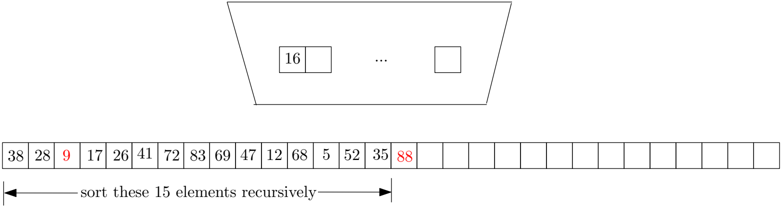
Let us consider that the subproblem has been solved. Now, the entire array is in ascending order. We thus finish the algorithm.

Example

Input:



After Step 2:



Analysis of Selection Sort

Let $f(n)$ be the maximum running time of selection sort when the problem size is n. We know:

$$f(1) = O(1)$$

For $n \geq 2$, we have:

$$f(n) \leq O(n) + f(n-1)$$

where the term $O(n)$ captures the cost of Steps 1 and 2, and $f(n-1)$ is the cost of Step 3.

Analysis of Selection Sort

So it remains to solve the recurrence (c_1, c_2 are constants):

$$\begin{aligned} f(1) &= c_1 \\ f(n) &\leq c_2 n + f(n-1) \end{aligned}$$

Using the expansion method, we get:

$$\begin{aligned} f(n) &\leq c_2 n + f(n-1) \\ &\leq c_2 n + c_2(n-1) + f(n-2) \\ &\leq c_2 n + c_2(n-1) + c_2(n-2) + f(n-3) \\ &\leq c_2 n + c_2(n-1) + \ldots + c_2 \cdot 2 + f(1) \\ &\leq c_2 n(n+1)/2 + c_1 \\ &= O(n^2). \end{aligned}$$

We now conclude that selection sort runs in $O(n^2)$ worst-case time.